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OPEN PROBLEMS

2nd International Conference on Fuzzy Sets Theory and Its Applications

During the workshops at the above conference, several open problems have been specified by the participants. We have decided to write them down and to publish them to give other colleagues of us an opportunity to think about the interesting and important problems posed by active researches in fuzzy logic. The problems vary in difficulty and they are divided into three groups. We wish all the possible solvers great success and satisfaction from their solution.

Radko Mesiar, Vilém Novák

t-norms

Problem 1. In [1, Theorem 14.2] it is proved that the range of a T_L-measure is convex and compact $\big(T_L(x,y) = \max(0, x+y-1)\big)$. Consider the family (T_s^F), $s \in [0,\infty]$, of fundamental t-norms (Frank's family).

Does the above Liapounoff type theorem hold for T_s^F-measures with $s \neq \infty$? Note that $T_\infty^F = T_L$.

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Erich P. Klement
klement@flll.uni-linz.ac.at

Problem 2. Let $P_1 = (\mathcal{P}_1, \mathcal{S}_1)$ and $P_2 = (\mathcal{P}_2, \mathcal{S}_2)$ be two fuzzy logics in the sense of [1] with the same set of atoms $\boldsymbol{P}$. P_1 is weaker than P_2 if there exists a function $f\colon \mathcal{F}(P_1) \to \mathcal{F}(P_2)$ such that for each formula $\mathcal{S} \in \mathcal{F}(P_1)$ and for each truth assignment $t\colon \boldsymbol{P} \to [0,1]$ we have $\bar{t}_{P_1}(\mathcal{S}) = \bar{t}_{P_2}\big(f(\mathcal{S})\big)$. Let (T_s^F), $s \in [0,\infty]$, be the family of fundamental t-norms (Frank's family).

Given an s-fuzzy logic P_s and an r-fuzzy logic P_r with $0 < r < s < \infty$, can we compare P_s and P_r (with respect to the relation "weaker")?

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Erich P. Klement
klement@flll.uni-linz.ac.at

Problem 3. Let T be a t-norm. Define the t-reverse T^* by

$$T^*(x,y) = \max\big(0, x+y-1+T(1-x,1-y)\big)\,.$$

In general, T^* need not be a t-norm (monotonicity and/or associativity may be violated). Call a t-norm T t-reversible if T^* is a t-norm, too.

Characterize the family of t-reversible t-norms.

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Erich P. Klement
klement@flll.uni-linz.ac.at

Problem 4. Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{F}(A)$ be the system of all $\mathcal{A}$-measurable fuzzy subsets of X (i.e., a generated tribe). A mapping $m\colon \mathcal{F}(\mathcal{A}) \to [0,\infty[$ is called a T-measure if it is left-continuous, $m(0)=0$ and $m(A\mathbf{T}B) + m(A\mathbf{S}B) = m(A) + m(B)$ for all $A, B \in \mathcal{F}(A)$, where T is a given t-norm (and the corresponding fuzzy intersection) and S is its dual t-conorm (and the corresponding fuzzy union).

Let T not be a fundamental t-norm (note that for fundamental t-norms, the characterization of T-measures is known [1]). Is then m necessarily a trivial T-measure, i.e., $m(A) = M(A>0)$, where M is some finite σ-additive measure on $\mathcal{A}$? Note that a trivial T-measure is a T-measure for each t-norm T without zero divizors.

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Radko Mesiar
mesiar@cvt.stuba.sk

Problem 5. Let $\{T_n\}$, $n \in \mathbb{N}$, be a family of continuous Archimedean t-norms such that $\lim T_n(x,y) = T(x,y)$, $x, y \in [0,1]$, where T is a continuous Archimedean t-norm.

Does there exist a family $\{f_n\}$, $n \in \mathbb{N}$, of additive generators of T_n such that there is an additive generator f of T and $\lim f_n(x) = f(x)$ for all $x \in [0,1]$?

Radko Mesiar
mesiar@cvt.stuba.sk

Problem 6. Let $M \subset [0,1]$ be a subset of the unit interval such that $0, 1 \in M$. Let T^* be binary operation on M such that all properties of a t-norm are fulfilled (i.e., T^* is associative, non-decreasing and commutative on M and 1 is the neutral element).

Characterize M and T^* such that there is a t-norm T extending T^* to whole $[0,1]$. If such T exists, when is it unique?

Radko Mesiar
mesiar@cvt.stuba.sk

Problem 7. By [1], the linearity of fuzzy intervals (i.e., triangular or trapezoidal shape) is preserved by an addition based on a t-norm T such that either T is weaker (or equal) than T_L, $T_L(x,y) = \max(0, x+y-1)$, or if T has an additive generator with strictly positive second derivative, then T is a member of the Yager's family of t-norms (T_p^Y), $p \in 1, \infty[$.

Characterize the family of t-norm T such that the addition of fuzzy intervals based on T preserves the linearity of fuzzy intervals.

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Radko Mesiar
mesiar@cvt.stuba.sk

Problem 8. Is a strictly monotone t-norm always continuous? M. Budinčević (Novi Sad) has given a negative answer to this problem. Namely, the t-norm T defined in the following way: for a fixed but arbitrary positive real number $k < 1$ we take $T(x,y) = kxy$, $x, y \in [0,1)$ and $T(x,1) = x$ for x on $[0,1]$ is not continuous.

Now there are three new questions:

(a) Does there exist a strictly monotone t-norm such that the continuity is broken inside (i.e., at some point $(x,y) \in (0,1)$)?

(b) Does there exist a strictly monotone t-norm which is a solution of the equation

$$T(kx, y) = T(x, ky), \; x, y \in [0,1)$$

for some fixed number $k \in (0,1)$ different from the t-norm $T(x,y) = sxy$, $s \in (0,1)$?

(c) Is a strictly monotone t-norm continuous in point $(1,1)$ always continuous?

Endre Pap
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
YU–21000 Novi Sad
YUGOSLAVIA

Problem 9. If T is Archimedean and $T(x,x)$, $x \in [0,1]$ is continuous, is T necessarily continuous on $[0,1] \times [0,1]$?
(This problem was stated in the book of Schweizer and Sklar PMS and Kimberling has given an example of continuous Archimedean t-norm which is not uniquely determined by its diagonal.)

Endre Pap

Problem 10. What is the minimal domain containing the main diagonal which determines a t-norm?
(Bezivin and Tomas (1993) have proven that strict t-norm is determined by values on both diagonals).

Endre Pap

Problem 11. A t-norm T_2 is dominated by a t-norm T_1, $T_1 \ll T_2$, if for each $a,b,c,d \in [0,1]$

$$T_2\big(T_1(a,b)\,,\ T_1(c,d)\big) \geq T_1\big(T_2(a,c)\,,\ T_2(b,d)\big)\,.$$

(The relation $\ll$ is reflexive and antisymmetric. For any t-norm we have $T_W \ll T \ll T_M$.)

Is the relation $\ll$ transitive?

Endre Pap

Problem 12. A surface is isothermal if its lines of curvature form an isothermally orthogonal net, i.e., if there exists a parametrization u, v such that lines curvature are the curves $u = \text{const}$ and $v = \text{const}$ and such that the first fundamental form is given by

$$ds^2 = \lambda(u,v)(du^2 + dv^2)\,.$$

Is every sufficiently smooth t-norm (associative surface) isothermal?

(This problem was stated in the book of Schweizer and Sklar PMS.)

Endre Pap

Problem 13. Characterize those triangular conorms S which satisfy

$$t = \sup\{e;\ e \le t\}, \qquad t \in [0,1],$$

where e is S dyadic number from $[0,1]$, i.e.,

$$e = \frac{a_1}{2}\, S\, \frac{a_2}{2^2}\, S \cdots S\, \frac{a_k}{2^k},$$

where $a_1, a_2, \ldots, a_k$ are zero or one.
(This problem occurs in n-dimensional Laypunoff convexity type theorem for the range of decomposable measures. For special t-conorms S_λ defined by

$$x S_\lambda y = \min\left\{(x^\lambda + y^\lambda)^{\frac{1}{\lambda}}, 1\right\},$$

where $0 < \lambda \le 1$ we have a positive answer.)

Endre Pap

Problem 14. Characterize those monotone set functions $m\colon \mathcal{F} \to [0,1]$ with $m(\emptyset) = 0$ for which there is a t-conorm S such that

$$m(A \cup B) = m(A) S\, m(B)$$

for $A \cap B \ne \emptyset$, $A, B \in \mathcal{F}$ (decomposable measures).
(There are monotone set functions which are decomposable with respect to any t-norm, for example $m_x(A) = 0$ for $x \notin A$ and $m_x(A) = 1$ for $x \in A$. However, there are monotone set functions which are not decomposable with respect to any t-conorm, for example: let $X = \{x, y\}$ and put $m(\{x\}) = 1/2$, $m(\{y\}) = 0$, $m(X) = 1$ and $m(\emptyset) = 0$.)

If you know that m is a decomposable measure, give an algorithm for finding the corresponding t-conorm S.

Endre Pap

Fuzzy logic

Problem 1. In fuzzy logic presented in [1, 2], it is possible to introduce additional n-ary connectives interpreted by corresponding n-ary operations $c\colon [0,1]^n \to [0,1]$. This fact makes this logic very wide. However, to keep the interesting properties of this logic and the properties of the additional connectives with respect to equivalence, the operations must be *logically fitting*, i.e., they must fulfil the following condition: There are non-zero natural numbers $k_1, \ldots, k_n$ such that

$$(a_1 \leftrightarrow b_1)^{k_1} \otimes \cdots \otimes (a_n \leftrightarrow b_n)^{k_n} \le c(a_1, \ldots, a_n) \leftrightarrow c(b_1, \ldots, b_n) \qquad (1)$$

holds for every $a_1, \dots, a_n$, $b_1, \dots, b_n \in [0,1]$, where the power is taken with respect to the Łukasiewicz product $\otimes$ and $\leftrightarrow$ is a biresiduation defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) = 1 - |a - b| .$$

Classify logically fitting *unary operations* which serve as a natural interpretation of linguistic modifiers and logically fitting t-norms and t-conorms. Note that every logically fitting operation is continuous and every Lipschitz continuous operation is logically fitting.

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Vilém Novák
novakv@osu.cz

Problem 2. The deduction theorem in the following form holds in fuzzy logic.

THEOREM 1. *Let A be a closed formula and $T' = T \cup \{\mathbf{1}/A\}$.*

(a) *If $T \vdash_a A^n \Rightarrow B$ and $T' \vdash_b B$ for some n then $a \leq b$.*

(b) *To every proof w' of B in T' there are an n and a proof w of $A^n \Rightarrow B$ in T such that*

$$\mathrm{Val}_T(w') = \mathrm{Val}_T(w).$$

Is it possible to prove that there is an n such that

$$T \vdash_a A^n \Rightarrow B \quad \text{iff} \quad T' \vdash_a B$$

holds true for $L = [0,1]$? (For L finite this holds.)

Vilém Novák
novakv@osu.cz

Problem 3. *Is Łukasiewicz's logic the best?* Maybe it is but one should notice the following:

(a) For 1-tautologies, both Łukasiewicz's and Gödel's propositional logic are axiomatizable, even decidable [1]. Łukasiewicz's predicate logic is not axiomatizable (i.e., the set of 1-tautologies is not recursively enumerable). It is unknown whether Gödel's predicate logic is axiomatizable.

(b) For graded provability, Łukasiewicz's propositional and predicate logic have Pavelka and Novák's completeness theorem [2, 3] and the obvious

analogue is impossible for Gödel's logic. However, on the one hand, the completeness theorems are highly ineffective and, on the other hand, possibly Gödel's logic admits a more general completeness theorem with comparative (non-quantitative) semantics of propositional constants for truth values.

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Petr Hájek
hajek@uivt.cas.cz

Problem 4. Consider Boolean model as a fuzzy set. By decomposing it into levels we obtain a family of crisp models. Now, we can consider a family of crisp models which satisfies the condition that it can abe synthetisized into an L-valued fuzzy set (that is that this family is a Moor's family of models under the inclusion). By synthesis of this family we obtain a fuzzy set (let us call it a fuzzy set L-valued model).

In another case, for arbitrary family of models there is an R-valued fuzzy set such that by decomposition into levels it gives this family (let us call this fuzzy set R-valued model).

The problem is to give logical interpretation of L and R-valued models. (Also we can choose some suitable families of models and investigate the structures obtained in that way).

Andreja Tepavčević
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
YU–21000 Novi Sad
YUGOSLAVIA

Problem 5. Any fuzzy set defined on an algebraic structure can be decomposed into the family of levels being crisp algebraic structures of the same type and having a lattice structure and vice versa, a Moore's family of substructures can be synthetisized into an L–valued fuzzy structure. Is there an algebraic property which is not preserved under the decomposition?

Branimir Šešelja
Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
YU–21000 Novi Sad
YUGOSLAVIA

Problem 6. Let $\{\mu_i\,;\, i \in I\}$ be a family of fuzzy sets on X and let $\{x_i\,;\, i \in I\}$ be a family of elements of X such that $\mu_i(x_i) = 1$ holds for every i.

THEOREM 1. *There exists an equality relation E w.r.t. the t-norm T (i.e., $E\colon X \times X \longrightarrow [0,1]$, $E(x,x) = 1$, $E(x,y) = E(y,x)$, $T\big(E(x,y), E(y,z)\big) \leq E(x,z)$ such that*

$$\mu_i(x) = E(x_i, x)\,, \qquad i \in I\,,\ x \in X\,,$$

iff

$$\sup_{x\in X} \big\{T\big(\mu_i(x)\,,\ \mu_j(x)\big)\big\} \leq \inf_{y\in X} \big\{\overleftrightarrow{T}\big(\mu_i(y)\,,\ \mu_j(y)\big)\big\} \qquad i\,,j \in I\,, \tag{1}$$

where

$$\overleftrightarrow{T}(\alpha,\beta) = \vec{T}\big(\max\{\alpha,\beta\}\,,\ \min\{\alpha,\beta\}\big)\,,$$

and

$$\vec{T}(\alpha,\beta) = \sup\big\{\gamma \in [0,1]\,;\ T(\alpha,\gamma) \leq \beta\big\}\,.$$

(1) is a necessary and sufficient condition for the existence of an equality relation such that the fuzzy sets μ_i are induced by the elements x_i. A sufficient condition is $T\big(\mu_i(x)\,,\ \mu_j(x)\big) = 0$ for all $i \neq j$. For a t-norm of the Yager's family this corresponds to $\big(1-\mu_i(x)\big)^p + \big(1-\mu_j(x)\big)^p \geq 1$. For $p > 0$ being very small this is a very weak condition.

Find a necessary (and sufficient) condition for the family of fuzzy sets such that there exists a lower semi-continuous t-norm T and an equality relation E w.r.t. T such that $\mu_i(x) = E(x_i, x)$.

Frank Klawonn
klawonn@ibr.cs.tu-bs.de

Problem 7. Interpreting fuzzy controllers on the basis of equality relations means that we have to solve the following problem. Given sets X, Y with equality relations E and F, respectively, and a mapping $\varphi_0\colon X_0 \to Y$, $X_0 \subseteq X$.

How can we construct an extensional extension of φ_0, i.e., $\varphi\colon X \to Y$, such that $\varphi|X_0 = \varphi_0$ and $E(x,x') \leq F\big(\varphi(x), \varphi(x')\big)$ for all $x, x' \in X$?

If E, F are equality relations w.r.t. the Łukasiewicz t-norm, the above question is equivalent to the following problem: Given to metric spaces (X,δ) and (Y,ρ) and a mapping $\varphi_0\colon X_0 \to Y$, $X_0 \subseteq X$. Is there an extension $\varphi\colon X \to Y$ of φ_0 such that

$$\delta(x,x') \leq 1 \implies \delta(x,x') \geq \rho\big(\varphi(x), \varphi(x')\big)\,?$$

Frank Klawonn
klawonn@ibr.cs.tu-bs.de

Problem 8. It is well known that fuzzy controllers of Mamdani or Sugeno type are universal approximators, i.e., they can approximate any continuous

function $\varphi\colon M \to \mathbb{R}$ with arbitrary precision (where $M \subseteq \mathbb{R}^n$ is compact). The proof is based on the Stone–Weierstraß theorem and therefore non-constructive. Klement et al. and Lee et al. have shown how a continuous function $\varphi[a,b] \to \mathbb{R}$ can be *exactly* realized by a fuzzy controller.

Find a similar construction for a function (perhaps with some additional assumptions) $\varphi[a,b]^n \to \mathbb{R}$ or find a *constructive* proof of an approximation theorem for such a function.

Frank Klawonn
klawonn@ibr.cs.tu-bs.de

Set representations of fuzzy quantum structures

Problem 1. A collection of fuzzy sets $F = [0,1]^X$ is called a *fuzzy quantum poset* (abbr. FQP) *of type I* (resp. *II*) if

(1) $\mathbf{1} \in F$,

(2) $a \in F \implies a' = \mathbf{1} - a \in F$,

(3) $\big[(a_i)_{i\in N} \in F,\ a_j \wedge a_k \le \mathbf{1}/2$ (resp. $a_j \le a'_k$) for $a_j \ne a_k\big] \implies \bigvee_{i\in N} a_i \in F$.

A *state* (P-measure) on F is a function $m\colon F \to [0,1]$ such that

(1) $m(a \vee a') = 1$,

(2) $m(\bigvee_{i\in N} a_i) = \sum_{i\in N} m(a_i)$ whenever $a_j \le a'_k$ for all $j \ne k$.

We define an equivalence relation $\sim$ on F by $a \sim b$ iff $m(a) = m(b)$ for all states m. If F is an FQP of type I, then $F\sim$ is a σ-orthomodular poset.

(a) Is there any analogy of this result for FQPs of type II ?

(b) Is every σ-orthomodular poset of the form $F\sim$ for some FQP F of type I ?

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Mirko Navara
navara@math.feld.cvut.cz

Problem 2. A d^3-lattice is a σ-complete lattice F with an order antiisomorphism $'$ such that

(d1) $b \wedge \bigvee_{i\in N} a_i = \bigvee_{i\in N} (b \wedge a_i)$,

(d2) $(a')' = a$,

(d3) $a \wedge a' \leq b \vee b'$.

States on d^3-lattices and the equivalence $\sim$ are defined as in the preceding problem. The quotient $F/\sim$ is a Boolean σ-algebra.

(a) Is (d3) necessary for this to hold?

(b) Is it sufficient to assume finite distributivity instead of (d1)?

(c) d^3-lattices generalize fuzzy quantum spaces (=FQPs which are lattices). Which d^3-lattices are isomorphic to fuzzy quantum spaces?

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Mirko Navara
navara@math.feld.cvut.cz