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SEMI–IMPLICATION ALGEBRA

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. An identity is normal if it is not of the form $x = p$, where p is a term different from the variable x. By a nilpotent shift of a variety $\mathcal{V}$ we mean a variety determined by all normal identities of $\mathcal{V}$. We study the nilpotent shift of the variety of all semilattices and that of the variety of implication algebras.

An identity is normal if it is not of the form $x = p$, where p is a term different from the variable x. One can associate with any variety $\mathcal{V}$ its *nilpotent shift* $\mathcal{N}(\mathcal{V})$ defined by all normal identities which are valid in $\mathcal{V}$. A variety $\mathcal{V}$ is *normally presented* if $\mathcal{V} = \mathcal{N}(\mathcal{V})$.

Since neither the variety $\mathcal{L}$ of all lattices nor the variety $\mathcal{D}$ of all distributive lattices are normally presented, we can find normally presented varieties containing $\mathcal{L}$ or $\mathcal{D}$; for such an attempt, see, e.g., [2], [3], [4] or [5]. Analogously, also the variety of all semilattices and the variety of all implication algebras are not normally presented. The aim of this paper is to give some natural generalizations of these varieties forming their nilpotent shift.

1. q-semilattices

A groupoid $(A, \cdot)$ is called a *q-semilattice* if it satisfies the following identities:

(associativity) $\quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
(commutativity) $\quad a \cdot b = b \cdot a$,
(weak idempotence) $\quad a \cdot (b \cdot b) = a \cdot b$.

Evidently, the variety of all q-semilattices is normally presented and it contains the variety of all semilattices. We are going to show the connection between

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a q-semilattice and a quasiorder on its support in the manner similar to that for semilattices and ordered sets.

By a *quasiorder* on a set A we mean a reflexive and transitive binary relation on A. If Q is a quasiorder on a set A and $a, b \in A$ we define

$$SU_Q(a,b) = \{z \in A;\ \langle a,z\rangle \in Q,\ \langle b,z\rangle \in Q \ \text{ and } \\ \text{if } \ \langle a,c\rangle \in Q\,,\ \langle b,c\rangle \in Q \ \text{ for some } c \in A \ \text{ then } \ \langle z,c\rangle \in Q\}\,.$$

Evidently, if Q is an order on A, then $\operatorname{card} SU_Q(a,b) \leq 1$ and

$$SU_Q(a,b) = \{\sup(a,b)\} \quad \text{if } \sup(a,b) \text{ exists}\,.$$

THEOREM 1. *Let $(A,\cdot)$ be a q-semilattice and Q be a binary relation on A introduced as follows*

$$\langle a,b\rangle \in Q \quad \textit{if and only if} \quad a\cdot b = b\cdot b\,.$$

Then Q is a quasiorder on A and $a\cdot b \in SU_Q(a,b)$ for every a, b of A.

Remark. This quasiorder Q will be called induced by the q-semilattice $(A,\cdot)$.

Proof. Reflexivity of Q is trivial. Prove transitivity of Q: let $\langle a,b\rangle \in Q$ and $\langle b,c\rangle \in Q$. Then

$$a\cdot b = b\cdot b \quad \text{and} \quad b\cdot c = c\cdot c\,,$$

and, by the q-semilattice identities, we obtain

$$a\cdot c = a\cdot(c\cdot c) = a\cdot(b\cdot c) = (a\cdot b)\cdot c = (b\cdot b)\cdot c = b\cdot c = c\cdot c\,,$$

whence $\langle a,c\rangle \in Q$.

Further, let $x, y \in A$. Then

$$x\cdot(x\cdot y) = (x\cdot x)\cdot y = x\cdot y\,,$$
$$y\cdot(x\cdot y) = x\cdot(y\cdot y) = x\cdot y\,,$$

thus $\langle x, x\cdot y\rangle \in Q$ and $\langle y, x\cdot y\rangle \in Q$. Suppose $\langle x,c\rangle \in Q$ and $\langle y,c\rangle \in Q$ for some $c \in A$. Then

$$x\cdot c = c\cdot c \quad \text{and} \quad y\cdot c = c\cdot c$$

which yield

$$(x\cdot y)\cdot c = (x\cdot y)\cdot(c\cdot c) = (x\cdot c)\cdot(y\cdot c) = (c\cdot c)\cdot(c\cdot c) = c\cdot c\,,$$

thus also $\langle x\cdot y, c\rangle \in Q$. We have proved

$$x\cdot y \in SU_Q(x,y)\,.$$

$\square$

R e m a r k . We can also introduce the relation Q^* on the support of a q-semilattice $(A, \cdot)$ as follows

$$\langle a, b\rangle \in Q^* \quad \text{if and only if} \quad a \cdot b = a \cdot a\,.$$

It is easy to see that also Q^* is a quasiorder on A. Moreover, Q^* is the inverse relation to the induced quasiorder Q, i.e., $Q^* = Q^{-1}$.

Let Q be a quasiorder on a set A. Then the relation $E_Q = Q \cap Q^{-1}$ is an equivalence on A and the factor relation Q/E_Q defined on the factor set A/E_Q by the rule

$$\big\langle [a]_{E_Q}, [b]_{E_Q}\big\rangle \in Q/E_Q \quad \text{if and only if} \quad \langle a, b\rangle \in Q\,,$$

is an order; if $\langle A/E_Q, Q/E_Q\rangle$ is a join-semilattice (with respect to the order Q/E_Q), then for every $[a]_{E_Q}, [b]_{E_Q} \in A/E_Q$ there exists $\sup\big([a]_{E_Q}, [b]_{E_Q}\big)$. Let κ be a choice function $\kappa\colon \exp A \to A$ such that $\kappa\big([a]_{E_Q}\big) \in [a]_{E_Q}$ for each $a \in A$. Introduce the operation $\vee$ on A as follows

$$a \vee b = \kappa\big(\sup([a]_{E_Q}, [b]_{E_Q})\big)\,.$$

It is easy to show that $(A, \vee)$ is a q-semilattice for every such a choice function (for details see [2]).

THEOREM 2. *Let Q be a quasiorder on a set A such that $SU_Q(a, b) \neq \emptyset$ for each $a, b \in A$. Then A can be equipped with a binary operation $\cdot$ such that $(A, \cdot)$ is a q-semilattice with*

(i) *$a \cdot b \in SU_Q(a, b)$ for each $a, b \in A$;*
(ii) *the quasiorder induced by $(A, \cdot)$ coincides with Q.*

P r o o f . Let κ be a choice function $\kappa\colon \exp A \to A$ with $\kappa\big(SU_Q(a, b)\big) \in SU_Q(a, b)$. Introduce the binary operation on A as follows

$$a \cdot b = \kappa\big(SU_Q(a, b)\big).$$

It is a routine way to testify associativity, commutativity and weak idempotence of $\cdot$. The condition (i) is trivial. For (ii), let $\langle a, b\rangle \in Q$. Then $SU_Q(a, b) = SU_Q(b, b)$ and hence $a \cdot b = \kappa\big(SU_Q(a, b)\big) = b \cdot b$. Conversely, if $a \cdot b = b \cdot b$ then

$$\kappa\big(SU_Q(a, b)\big) = \kappa\big(SU_Q(b, b)\big)\,,$$

which implies $\langle a, b\rangle \in Q$. Therefore, Q coincides with the quasiorder induced by $(A, \cdot)$. □

A finite quasiordered set A can be visualized by a diagram: elements of A are represented by points in a plane which are connected by arrows. The symbol

means $\langle a, b\rangle \in Q$ and Q is the reflexive and transitive hull of the relation visualized by arrows.

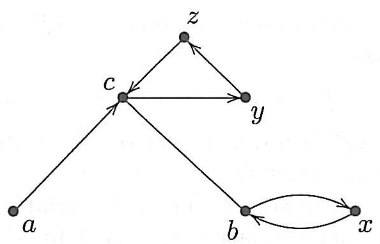

FIGURE 1.

EXAMPLE 1. Let A be a quasiordered set visualized by the diagram in Fig. 1. Evidently, $SU_Q(a,b) = \{c,y,z\}$, $SU_Q(b,b) = SU_Q(x,x) = SU_Q(b,x) = \{b,x\}$, $SU_Q(a,c) = SU_Q(b,c) = SU_Q(x,c) = SU_Q(y,c) = SU_Q(z,c) = SU_Q(a,y) = SU_Q(a,z) = SU_Q(b,y) = SU_Q(b,z) = SU_Q(z,y) = SU_Q(x,y) = SU_Q(x,z) = \{c,y,z\}$, $SU_Q(a,a) = \{a\}$.

By Theorem 2, A can became a q-semilattice; we can choose, e.g., $\kappa\big(SU_Q(a,b)\big) = c$, $\kappa\big(SU_Q(b,b)\big) = b$, i.e., $a \cdot b = c$, $b \cdot b = b$ etc., the operation table is the following:

$\cdot$	a	b	c	x	y	z
a	a	c	c	c	c	c
b	c	b	c	b	c	c
c	c	c	c	c	c	c
x	c	b	c	b	c	c
y	c	c	c	c	c	c
z	c	c	c	c	c	c

For another choice function, the operation $\cdot$ would be different but the induced quasiorder is the same.

Remark. There exist two non-isomorphic two-element q-semilattices, namely, those of S_1, S_2 in Fig. 2:

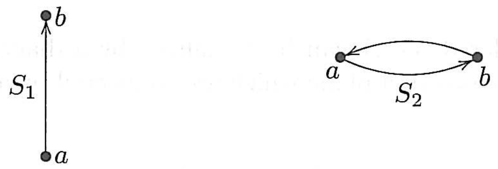

FIGURE 2.

THEOREM 3. *The variety $\mathcal{S}$ of all q-semilattices is locally small; the free q-semilattice $F_{\mathcal{S}}(x_1,\dots,x_n)$ with n free generators has exactly 2^n-1+n elements.*

Proof. Let $x_1,\dots,x_n$ be different free generators of the free q-semilattice $F_{\mathcal{S}}(x_1,\dots,x_n)$. Then $x_i \neq x_i \cdot x_i$, but $x_i \cdot x_j = x_i \cdot (x_j \cdot x_j)$.

Hence $F_{\mathcal{S}}(x_1,\dots,x_n)$ contains a free semilattice generated by $x_1,\dots,x_n$ and with every x_i also the element $x_i \cdot x_i$ satisfying

$$\langle x_i, x_i \cdot x_i\rangle \in Q \quad \text{and} \quad \langle x_i \cdot x_i, x_i\rangle \in Q$$

as can be easily shown (Q is the induced quasiorder). Hence it contains n elements $x_1 \cdot x_1,\dots,x_n \cdot x_n$ and all of the elements of the free semilattice which has exactly 2^n-1 elements. □

EXAMPLE 2. The free q-semilattices with one, two and three generators are depicted in Fig. 3.

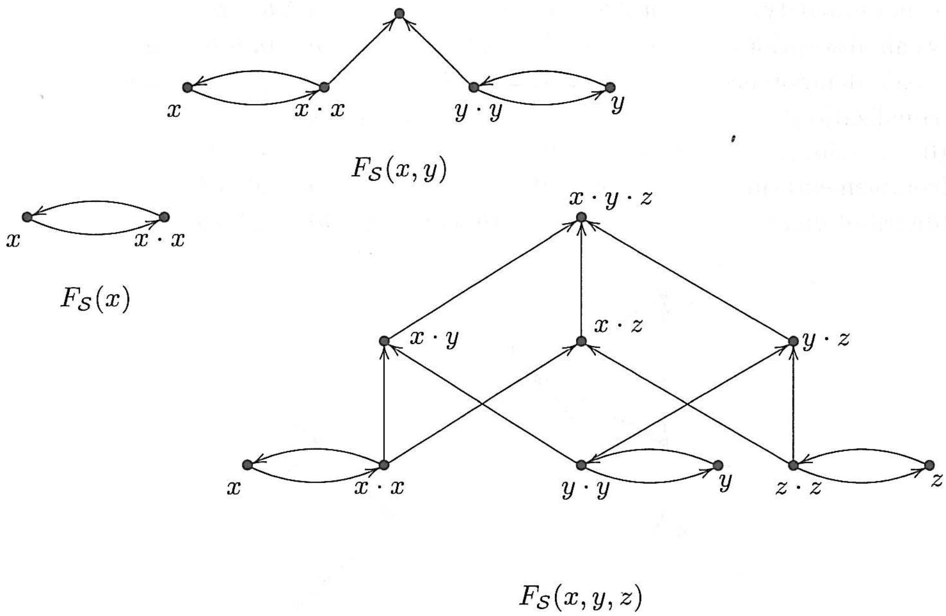

FIGURE 3.

An element b of a q-semilattice $(A,\cdot)$ is called an *idempotent* if $b \cdot b = b$. Denote by S_A the set of all idempotents of $(A,\cdot)$, the so-called *skeleton*. It is almost obvious that $(S_A,\cdot)$ is a semilattice and it is the greatest sub-q-semilattice of $(A,\cdot)$ which is a semilattice. Henceforth, the restriction $Q|S_A$ of the induced quasiorder Q onto S_A is an order.

For a q-semilattice $(A,\cdot)$, a subset $C \subseteq A$ is called a *cell of* $(A,\cdot)$ provided $\operatorname{card} C > 1$, $a \cdot a = b \cdot b$ for each $a, b \in C$ and C is maximal with respect to this property.

It is easy to see that every q-semilattice $(A,\cdot)$ consists of its skeleton S_A and a system $\{C_\gamma;\ \gamma \in \Gamma\}$ of cells and, moreover, $\operatorname{card}(C_\gamma \cap S_A) = 1$ for each $\gamma \in \Gamma$, i.e., every cell has exactly one idempotent.

In Example 1, elements $\{a, b, c\}$ form the skeleton S_A and $(A,\cdot)$ has two cells, namely, $C_1 = \{b, x\}$ with the idempotent b, and $C_2 = \{c, y, z\}$ with the idempotent c. In Example 2, the skeleton of $F_S(x, y, z)$ coincides with the free semilattice with three free generators x, y, z and it contains three cells: $C_1 = \{x, x \cdot x\}$, $C_2 = \{y, y \cdot y\}$, $C_3 = \{z, z \cdot z\}$.

For some reasons of the next section, let us recall the concept of q-algebra (see [2], [3] or [4]): An algebra $(A; \vee, \wedge, ', 0, 1)$ of the type $(2, 2, 1, 0, 0)$ is called a *q-algebra* if it satisfies the following identities:

(associativity)	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
(commutativity)	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
(weak absorption)	$a \vee (a \wedge b) = a \vee a$	$a \wedge (a \vee b) = a \vee a$
(weak idempotence)	$a \vee (b \vee b) = a \vee b$	$a \wedge (b \wedge b) = a \wedge b$
(equalization)	$a \vee a = a \wedge a$	
$(0-1$ axioms)	$a \wedge 0 = 0$	$a \vee 1 = 1$
(complementation)	$a \wedge a' = 0$	$a \vee a' = 1$
(distributivity)	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	

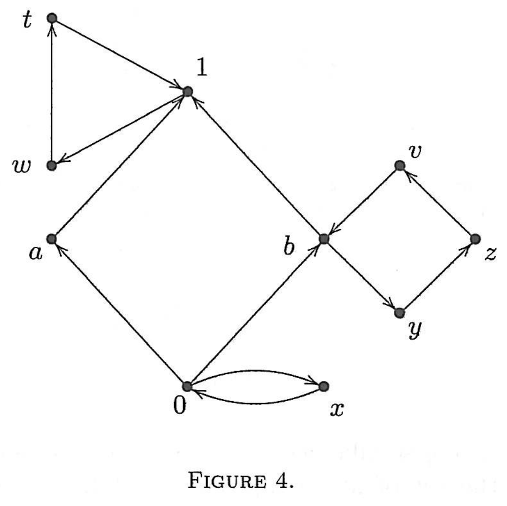

FIGURE 4.

It is almost obvious that if $(A; \vee, \wedge, ', 0, 1)$ is a q-algebra, then $(A, \vee)$ and $(A, \wedge)$ are q-semilattices. Hence, there exists an induced quasiorder Q on A

such that $a \vee b \in SU_Q(a,b)$ and $a \wedge b \in SU_{Q^{-1}}(a,b)$. Moreover, $\langle 0,a\rangle \in Q$ and $\langle a,1\rangle \in Q$ for each $a \in A$. Hence, every q-algebra can be depicted by a diagram (introduced for quasiordered sets). Elements 0 and 1 are clearly idempotents of every q-algebra. Also the concepts of skeleton S_A and cell C are the same as for q-semilattices. For some details see [2], [3] or [4]. An example of a q-algebra which is not a Boolean algebra is shown in Fig. 4.

Its skeleton is $S_A = \{0,a,b,1\}$, it has three cells. Moreover, $a' = b$, $b' = a$ and $y' = a$, $z' = a$, $v' = a$, $1' = t' = 0 = w'$, $x' = 0' = 1$.

Hence, every Boolean algebra is a q-algebra but not vice versa. Moreover, the variety of all q-algebras is normally presented.

2. Semi-implication algebras

The concept of implication algebras was introduced by J. C. Abbott [1]. Since every implication algebra is a semilattice with respect to the term operation, it motivates our effort to generalize this concept in order to obtain an algebra which is a q-semilattice with respect to the analogous term operation.

A groupoid $(A,\cdot)$ is called a *semi-implication algebra* if it satisfies the following identities

1° $z\big[(xy)x\big] = zx$ and $\big[(xy)x\big]z = xz$,

2° $(xy)y = (yx)x$,

3° $x(yz) = y(xz)$

(where the binary operation $\cdot$ will be expressed by juxtaposition only).

Note that the identities 2° and 3° coincide with those for implication algebra and the axiom $(xy)x = x$ for implication algebra is replaced by 1°. Hence, every implication algebra is a semi-implication algebra but not vice versa. Moreover, the variety of all semi-implication algebras is normally presented and it is equal to the nilpotent shift of the variety of all implication algebras.

We are going to list basic properties of semi-implication algebra:

Theorem 4. *Every semi-implication algebra $(A,\cdot)$ contains a nullary term operation 1 and satisfies the following identities:*

4° $aa = 1$, $1(1a) = 1a$, $a1 = 1$;

5° $(1a)b = a(1b) = (1a)(1b) = 1(ab) = ab$;

6° $a(ab) = ab$, $a(ba) = 1$,
$a\big[(ab)b\big] = 1$, $(ab)(ba) = ba$,
$\big[(ab)b\big]b = ab$, $\big[(ab)b\big]a = ba$.

If $ab = 1$ and $ba = 1$ for some $a,b \in A$, then $1a = 1b$.

Proof. For every x, y of A we can derive by 1°, 2°, 3°:

$$xx = [(xy)x]x = [x(xy)](xy) = x[[x(xy)]y] = x[[((xy)x)(xy)]y] = \\ = x[(xy)y] = (xy)(xy),$$

i.e., $(A, \cdot)$ satisfies the identity

(e) $xx = (xy)(xy)$.

By (e) and 2° we conclude

$$xx = (xy)(xy) = [(xy)y][(xy)y] = [(yx)x][(yx)x] = (yx)(yx) = yy,$$

thus xx is a term nullary operation; denote it by 1.

Now, $a(1b) = a((bb)b) = ab$ by 1° and the foregoing identity. Analogously,

$$(1a)b = ((aa))b = ab \quad \text{and, similarly,} \quad (1a)(1b) = ab.$$

Moreover, by 1° and 3° we infer $1(ab) = a((bb)b) = ab$ proving the remaining identity of 5°. The second identity of 4°, namely, $1(1a) = 1a$, is a trivial consequence of 5°.

Further,

$$a(ab) = [(ab)a](ab) = 1[[(ab)a](ab)] = 1(ab) = ab$$

proving the first identity of 6°. Hence, also $a1 = a(aa) = aa = 1$ proving 4°. For remaining identities of 6°, we can count:

$$a(ba) = b(aa) = b1 = 1,$$
$$a[(ab)b] = (ab)(ab) = 1,$$
$$(ab)(ba) = b[(ab)a] = ba,$$
$$[(ab)b]b = [b(ab)](ab) = [a(bb)](ab) = (a1)(ab) = 1(ab) = ab,$$
$$[(ab)b]a = [(ba)a]a = ba \quad \text{by the previous identity}.$$

Finally, if $ab = 1$ and $ba = 1$ for some $a, b \in A$, then

$$1a = (ba)a = (ab)b = 1b.$$

□

THEOREM 5. *Let $(A, \cdot)$ be a semi-implication algebra. The binary relation Q on A introduced as follows*

$$\langle a, b\rangle \in Q \quad \text{if and only if} \quad (ab)b = 1b$$

is a quasiorder. Moreover, $(A, \vee)$ is a q-semilattice for the term operation $\vee$ defined by

$$a \vee b = (ab)b$$

and Q coincides with the quasiorder induced by $(A,\vee)$.

For each $a\in A$ it holds $\langle a,1\rangle\in Q$. If $\langle 1,a\rangle\in Q$ for some $a\in A$, then $1a=1$.

P r o o f. At first, we prove that Q is a quasiorder on A. Since

$(aa)a=1a$,

we infer reflexivity of Q. Suppose $\langle a,b\rangle\in Q$ and $\langle b,c\rangle\in Q$. Then

$$(ab)b=1b \quad\text{and}\quad (bc)c=1c,$$

thus

$$(ac)c=\big[a(1c)\big]c=\big[a[(bc)c]\big]c=\big[a[(cb)b]\big]c=\big[(cb)(ab)\big]c=\big[(cb)1\big]c=1c,$$

(since $ab=a(1b)=a\big[(ab)b\big]=1$) giving $\langle a,c\rangle\in Q$, thus Q is a quasiorder on A.

By 4°, we have $(a1)1=11$ proving $\langle a,1\rangle\in Q$ for each $a\in A$. Further, if $\langle 1,a\rangle\in Q$ for some $a\in A$, then $(1a)a=1a$ but

$(1a)a=aa=1$

applying 5° and 4°, i.e., $1a=1$.

With respect to Theorem 1, it satisfies to prove that

$(ab)b\in SU_Q(a,b)$.

If $ab=1$ then $(ab)b=1b$, i.e., $\langle a,b\rangle\in Q$. This implies, by

$$a\big[(ab)b\big]=(ab)(ab)=1,$$

the relation $\langle a,(ab)b\rangle\in Q$ for each $a,b\in A$ and also $\langle b,(ab)b\rangle=\langle b,(ba)a\rangle\in Q$. If, moreover, $\langle a,c\rangle\in Q$ and $\langle b,c\rangle\in Q$ for some $c\in A$, then

$(ac)c=1c$ and

$$1c=(bc)c=(cb)b=xb \quad\text{for}\quad x=cb$$

thus

$$\begin{aligned}([(ab)b]c)c&=([(ab)b](1c))c=([(ab)b](xb))c=(x([(ab)b]b))c=\\&=(x[(b(ab))(ab)])c=(x[[a(bb)](ab)])c=\\&=(x[(a1)(ab)])c=((x(1(ab)))c=(a(xb))c=(a(1c))c=\\&=(ac)c=1c,\end{aligned}$$

whence $\langle (ab)b,c\rangle\in Q$ proving $(ab)b\in SU_Q(a,b)$. By Theorem 2, $(A,\vee)$ is a q-semilattice for $a\vee b=(ab)b$. □

COROLLARY. *The free semi-implication algebra with one free generator has 3 elements. The free semi-implication algebra with two free generators has 8 elements. Their diagrams as q-semilattices are depicted in Fig. 5 and the operation table is the following:*

	1	x	$1x$	y	$1y$	xy	yx	$(xy)y$
1	1	$1x$	$1x$	$1y$	$1y$	xy	yx	$(xy)y$
x	1	1	1	xy	xy	xy	1	1
$1x$	1	1	1	xy	xy	xy	1	1
y	1	yx	yx	1	1	1	yx	1
$1y$	1	yx	yx	1	1	1	yx	1
xy	1	$1x$	$1x$	$(xy)y$	$(xy)y$	1	yx	$(xy)y$
yx	1	$(xy)y$	$(xy)y$	$1y$	$1y$	xy	1	$(xy)y$
$(xy)y$	1	yx	yx	xy	xy	xy	yx	1

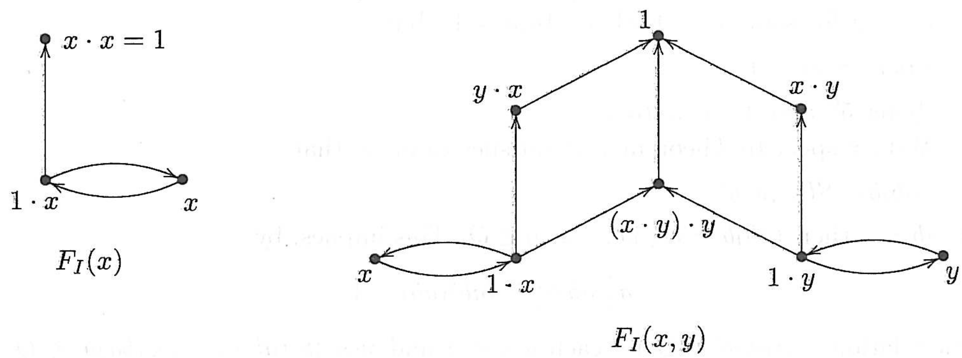

FIGURE 5.

The proof follows directly by Theorem 4 and 5. □

R e m a r k . For an implication algebra $(A, \cdot)$, the q-semilattice $(A, \vee)$ with the operation $a \vee b = (ab)b$ will be called the induced q-semilattice.

THEOREM 6. *Let $(A, \cdot)$ be a semi-implication algebra and $(A, \vee)$ its induced q-semilattice.*

(1) *$a \in A$ is an idempotent of $(A, \vee)$ if and only if $a = 1a$ in $(A, \cdot)$;*

(2) *elements $a, b \in A$ are of the same cell of $(A, \vee)$ if and only if $1a = 1b$ in $(A, \cdot)$;*

(3) *if S_A is the skeleton of $(A, \vee)$, then $(S_A, \cdot)$ is the greatest implication algebra contained in $(A, \cdot)$.*

Proof. (1) follows immediately from the fact that $a \vee a = (aa)a = 1a$.
(2): Elements $a, b \in A$ are of the same cell if and only if $\langle a, b\rangle \in Q$ and $\langle b, a\rangle \in Q$, which is equivalent to $(ab)b = 1b$ and $(ba)a = 1a$; by 2°, we have $1b = (ab)b = (ba)a = 1a$.

(3): Let $x, y \in S_A$, i.e., they are idempotents of $(A, \vee)$. Then, by (1), $x = 1x$ and $y = 1y$. By using of 5°, we have

$$(xy)x = 1\big[(xy)x\big] = 1x = x\,,$$

i.e., the elements of S_A satisfy the remaining axiom of implication algebra.

Moreover, if $x, y \in A$ satisfy $(xy)x = x$ then $x = (xy)x = 1[(xy)x] = 1x$ and, by (1), $x \in S_A$. □

Since $(S_A, \vee)$ is a join-semilattice, the restriction of Q onto S_A is an order. Therefore, it is meaningful to speak about suprema or infima of elements of the skeleton S_A.

Theorem 7. *Let $(A, \cdot)$ be a semi-implication algebra.*

(1) *If $a, b \in S_A$ and p is any lower bound of a, b, then*

$$\inf(a, b) = \big[a(bp)\big]p\,;$$

(2) *If $p \in A$ then the set*

$$B_p = \{a \in A;\ \langle p, a\rangle \in Q\}$$

is the q-algebra $(B_p; \vee, \wedge, ', c, 1)$, where $a \vee b = (ab)b$, $a \wedge b = \big[a(bp)\big]p$, $a' = a \cdot p$, and c is the idempotent of the cell containing p.

The proof is an immediate consequence of Theorems 5 and 6 in [1] and the foregoing Theorem 6.

Corollary. *Every semi-implication algebra $(A, \cdot)$ is a q-semilattice in which for any $p \in A$ the set B_p is a q-algebra.*

Remark. Although the most of proven properties of semi-implication algebras are analogous to those of implication algebras, there are also important differences. For example, the variety of all implication algebras is congruence 3-permutable, i.e.,

$$\theta \circ \Phi \circ \theta = \Phi \circ \theta \circ \Phi$$

for every two congruences θ, Φ on $(A, \cdot)$. On the other hand, the variety of all semi-implication algebras is not congruence n-permutable for any integer n. The following example shows that the free semi-implication algebra $F_I(x, y)$ is not congruence 4-permutable.

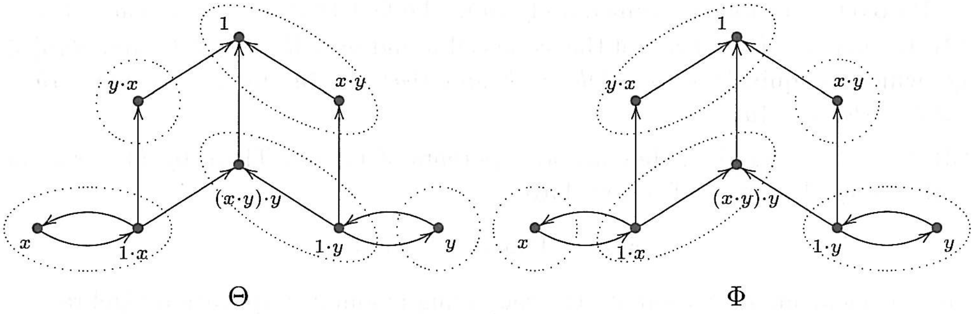

FIGURE 6.

EXAMPLE 4. Let θ, Φ be congruences on $F_I(x,y)$ determined by their classes in Fig. 6.

Then $\langle x,y\rangle \in \theta \circ \Phi \circ \theta \circ \phi$ but $\langle x,y\rangle \notin \Phi \circ \theta \circ \Phi \circ \theta$.

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