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ON LOWER–SEMICONTINUITY OF VARIATIONAL INTEGRALS

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ABSTRACT. We consider questions regarding the relation between Morrey's quasiconvexity condition and lower semicontinuity of variational integrals. We also show how quasiconvexity naturally comes up in problems concerning stability of sets of solutions of certain nonlinear systems under weak convergence.

I. Lower-semicontinuity and quasiconvexity

We consider variational integrals

$$I(u) = \int_\Omega f\big(Du(x)\big)\,dx$$

defined for (sufficiently regular) functions $u\colon \Omega \to \mathbb{R}^m$. Here Ω is a bounded open subset of $\mathbb{R}^n$, $Du(x)$ denotes the gradient matrix of u at x and f is a continuous real valued function on the space of all real $m \times n$ matrices $\mathscr{M}^{m\times n}$.

One of the important questions in the Calculus of Variations is the following: for which functions f is the integral I weakly lower-semicontinuous in the following sense: $I(u) \le \liminf_{j\to\infty} I(u_j)$ for every sequence of functions $u_j : \Omega \to \mathbb{R}^m$ satisfying $\big|Du_j(x)\big| \le c$ (for some $c \ge 0$) and converging (locally) uniformly in Ω to $u\colon \Omega \to \mathbb{R}^m$.

This question has been studied in [Mo1] (see also [Mo2]) where the following notion was introduced. We say that f is *quasiconvex* if for any matrix $\mathbb{A} \in \mathscr{M}^{m\times n}$ and any smooth function $\varphi\colon \Omega \to \mathbb{R}^m$ compactly supported in Ω the inequality $\int_\Omega f(\mathbb{A} + D\varphi)\,dx \ge \int_\Omega f(\mathbb{A})\,dx$ holds.

The class of quasicovex functions is independent of Ω. (See [Mo1], [Mo2].)

In [Mo1] (see also [Mo2]) the following result has been proved:

THEOREM. (**Morrey**). *I is weakly lower-semicontinuous in the above sense if and only if f is quasiconvex.*

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We remark that under natural growth assumptions quasiconvexity is also a necessary and sufficient condition for the weak sequential lower-semicontinuity of I on $W^{1,p}$ spaces. See [AF1] for optimal results in this direction.

The quasiconvexity condition plays also an important role in results regarding partial regularity of minimizers of the integral I, see [Ev], [AF2].

It is not difficult to verify that for $n = 1$ or $m = 1$ quasiconvexity reduces to convexity. On the other hand, for $n \geq 2$ and $m \geq 2$ there always exist nonconvex quasiconvex functions. (A typical example in the case $m = n$ is $f(\mathbf{X}) = \det \mathbf{X}$). In fact, it turns out that it may be very difficult to decide whether or not a given function is quasiconvex. For specific examples see [AD], [DM], [Sv1]. In this connection, the following simpler notions have been introduced, see [Ba1], [Mo1], [Mo2]:

f is *rank-one convex* if for each matrix $\mathbf{A} \in \mathscr{M}^{m\times n}$ and each rank-one matrix $\mathbf{B} \in \mathscr{M}^{m\times n}$ the function $t \to f(\mathbf{A} + t\mathbf{B})$ is convex. (For C^2–functions rank-one convexity is exactly the same as the so-called *Legendre–Hadamard condition*, see [Ba1].)

f is *polyconvex* if $f(\mathbf{X})$ = convex function of minors of the matrix $\mathbf{X}$. (For example, $f\colon \mathscr{M}^{2\times 2} \to \mathbb{R}$ is polyconvex if there exists a convex function $G\colon \mathscr{M}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $f(\mathbf{X}) = G(\mathbf{X}, \det \mathbf{X})$ for each $\mathbf{X} \in \mathscr{M}^{2\times 2}$.)

It is well-known that rank-one convexity (RC) is a necessary condition for quasiconvexity (QC) and that polyconvexity (PC) is a sufficient condition for quasiconvexity. In other words, PC$\Rightarrow$QC$\Rightarrow$RC. We remark that in principle it should be relatively easy to decide whether or not a given function is rank-one convex or polyconvex (although actual computations can be lengthy and tedious). It is therefore of great interest to know whether or not there are further relations between the three notions of convexity introduced above.

It turns out that there are quasiconvex functions which are not polyconvex, see [Te], [Se], [Ba2], [AD], [Sv2].

For a long time it was a major open problem whether or not RC$\Rightarrow$QC. It turns out that for $n \geq 2$, $m \geq 3$ this fails, see [Sv4] where an example is given which shows that for $n \geq 2$, $m \geq 3$ there exists a fourth-order polynomial which is rank-one convex but not quasiconvex. The case $n \geq 2$, $m = 2$ remains open. We remark that even in the case $n = m = 2$ the implication RC$\Rightarrow$QC would have far-reaching consequences.

II. Quasiconvexity and Compensated Compactness

Let $\mathcal{K} \subset \mathscr{M}^{m\times n}$ be a closed set. We define the following "semi-convex" hulls of $\mathcal{K}$. The *quasiconvex hull*, $\mathcal{K}^{qc}$, is defined by:

$$\mathbf{X} \notin \mathcal{K}^{qc} \quad \text{if and only if} \quad f(\mathbf{X}) > \sup_{\mathcal{K}} f \quad \text{for some quasiconvex function } f\,.$$

The rank-one convex hull $\mathcal{K}^{rc}$ and the polyconvex hull $\mathcal{K}^{pc}$ are defined in a similar way by replacing the class of quasiconvex functions in the last definition by the class of rank-one convex functions and polyconvex fuctions respectively.

It turns out that the problem of computing $\mathcal{K}^{qc}$ for a given set $\mathcal{K}$ arises in many different situations. As an example, let us consider the following problem from the theory of Compensated Compactness. (For an exposition of the theory of Compensated Compactness see [Ta].)

Let $\mathcal{K} \subset \mathscr{M}^{m\times n}$ be a closed set and let us consider the following first-order system of PDE for functions $u\colon \Omega \to \mathbb{R}^m$.

$$Du(x) \in \mathcal{K}. \tag{S}$$

We say that the system (S) is *strongly stable* if for each sequence of uniformly Lipschitzian functions $u_j\colon \Omega \to \mathbb{R}^m$ (that is $|Du_j(x)| \le c$ for some $c > 0$) which converges uniformly to a function $u\colon \Omega \to \mathbb{R}^m$ and satisfies $\lim\limits_{j\to\infty} \int\limits_{\Omega} \operatorname{dist}\big(Du_j(x), \mathcal{K}\big)\, dx = 0$, the function u solves the system (S), i.e., $Du(x) \in \mathcal{K}$ for a.e. $x \in \Omega$.

One of the main problems in the theory of Compensated Compactness is the problem of classifying the strongly stable systems. It turns out that this is very closely related to the problem of classifying the quasiconvex functions. We have the following:

Proposition. *In the notation introduced above, assume that $\mathcal{K}$ is compact. Then the system* (S) *is strongly stable if and only if $\mathcal{K}^{qc} = \mathcal{K}$.*

This result is quite simple (and it is an easy exercise to prove it), but it seems to be helpful in understanding the nature of the problem of classifying the strongly stable systems.

An obvious consequence of the Proposition and the fact that QC$\Rightarrow$RC is the following: *a necessary condition for* (S) *to be strongly stable is that $\mathcal{K}$ satisfies $\mathcal{K}^{rc} = \mathcal{K}$*. We remark that although it is believed that this condition is not sufficient for the strong stability of (S), no counterexample is known which would confirm this.

Another area where the problem of computing $\mathcal{K}^{qc}$ comes up is the theory of microsructures recently developed in [BJ1]. See also [BJ2] and [Sv6].

The problem of computing $\mathcal{K}^{qc}$ for a given set $\mathcal{K}$ is in general very difficult and in many cases seems to be out of reach of the present methods.

Example. Let $m = n = 2$ and let $\mathbb{A} = \mathbb{0}$, $\mathbb{B} = \mathbb{I}$, $\mathbb{C} = \operatorname{diag}(c_1, c_2)$ with $c_1 > 1$ and $0 < c_2 < 1$. Let $\mathcal{K} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}\} \subset \mathscr{M}^{2\times 2}$. It can be proved that in this case $\mathcal{K}^{qc} = \mathcal{K}$, see [Sv3] and [Sv5]. I do not know any simple proof of this statement. (It is not difficult to see that in this example we have $\mathcal{K}^{pc} \ne \mathcal{K}$).

In general it can be proved that for any set $\mathcal{K} \subset \mathscr{M}^{m\times n}$ consisting of three matrixes no two of which are rank-one connected we have $\mathcal{K}^{qc} = \mathcal{K}$.

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