

ON THE h , p AND $h - p$ VERSION OF THE FINITE ELEMENT METHOD

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ABSTRACT. The paper surveys the basic approximation properties of the h, p and $h - p$ versions of the finite element method.

1. Introduction

A major tool in today's computational mechanics is the finite element method (FEM). This method has a long history (see [B2][O] and references there). The roots of this method are in engineering structural mechanics. The major progress in FEM as a tools in computational engineering started in 1960 and in mathematics after 1970. Today the finite element method is used very widely in many fields, in structural (solid) mechanics, fluid mechanics, thermal analysis, electrical engineering, etc. for both linear and nonlinear problems, stationary and transient analyses. Consequently, there is a vast body of published research. From 1975 until today about 50,000 papers in solid mechanics are in the MAKEBASE database [M1], [M2].

Most of the papers on finite elements are related to the classical form of FEM, called *the h -version* (see Sec. 2 for definition). There are many widely available commercial and research codes based on the h -version (e.g. MSC/NASTRAN, ANSYS, ADINA, ...).

Relatively recently the so called p and $h - p$ version (see Sec. 2) was developed and today a few successful commercial and large research programs are available (APPLIED STRUCTURE, MSC/PROBE, PHLEX, STRIPE, Poly-FEM, PEGASYS).

The first theoretical papers on the p and $h - p$ versions ([BSK], [BD]) were published in 1981. Many books about the h -version in engineering and mathematics are available. In contrast the only book addressing the p and $h - p$ version appeared in 1991 [SzB].

The $h - p$ version of FEM has various theoretical, computational, implementational and engineering features which are quite different when compared

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with the classical h -version. In this paper we will address only a few of these, especially those of a theoretical nature. We will focus on the problems of the approximation, the rate of convergence and some comparison of the h , p and $h-p$ version of FEM. We will also bring an illustrative example. For the survey of various aspects of the p and $h-p$ version we refer to [AB], [BG5], [BS4] (see also references there).

2. The finite element method (FEM)

Today (for elliptic problems) the finite element method treats the problems formulated in a variational (weak) form.

Given two reflexive Banach spaces H_1 and H_2 and the bilinear form $B(u, v)$ defined on $H_1 \times H_2$ we seek $u \in H_1$ such that

$$B(u, v) = F(v), \quad \forall v \in H_2, \quad (2.1)$$

where $F \in H_2'$ (i.e., F is linear functional on H_2).

About the bilinear form we will assume the following:

$$|B(u, v)| \leq C(H_1, H_2) \|u\|_{H_1} \|v\|_{H_2}, \quad (2.2)$$

$$\inf_{\substack{u \in H_1 \\ \|u\|_{H_1}=1}} \sup_{\substack{v \in H_2 \\ \|v\|_{H_2}=1}} |B(u, v)| = \gamma(H_1, H_2) > 0, \quad (2.3)$$

$$\sup_{\substack{u \in H_1 \\ \|u\|_{H_1}=1}} |B(u, v)| > 0, \quad \forall v \in H_2, v \neq 0. \quad (2.4)$$

We have the following theorem

THEOREM 2.1. [B1] [BA1]. *Let $F \in H_2'$,*

$$\|F\|_{H_2'} = \sup_{\|v\|_{H_2}=1} F(v), \quad (2.5)$$

and $B(u, v)$ satisfies (2.2)–(2.4). Then there exists unique solution $u_0 \in H_1$ satisfying (2.1) and

$$\|u_0\|_{H_1} \leq \frac{1}{\gamma(H_1, H_2)} \|F\|_{H_2'}. \quad (2.6)$$

Theorem (2.1) obviously shows an isomorphism between H_1 and H_2' .

What is finite element method?

Let $S_i(h) \subset H_i$, $i = 1, 2$, $0 < h < 1$ be a one parameter family $F(S_1, S_2)$ of finite dimensional subspaces. (The finite element utilizes special subspaces S_i .)

The finite element method seeks $u_{S_1} \in S_1$ such that

$$B(u_{S_1}, v) = F(v), \quad \forall v \in S_2. \quad (2.7)$$

We will assume additionally that

$$\inf_{\substack{u \in S_1 \\ \|u\|_{H_1}=1}} \sup_{\substack{v \in S_2 \\ \|v\|_{H_2}=1}} |B(u, v)| \geq J(S_1, S_2) > 0, \quad (2.8)$$

$$\sup_{\substack{u \in S_1 \\ \|u\|_{H_1}=1}} |B(u, v)| > 0, \quad v \in S_2, v \neq 0. \quad (2.9)$$

Then there exists the unique solution $u_{S_1} \in S_1$ and

THEOREM 2.2. [B1] [BA1]. *We have*

$$\|u_{S_1} - u_0\|_{H_1} \leq D(H_1, H_2, S_1, S_2) \inf_{\chi \in S_1} \|u - \chi\|_{H_1}, \quad (2.10)$$

where u_0 is the solution of the problem (2.1) and

$$D(H_1, H_2, S_1, S_2) = 1 + \frac{C(H_1, H_2)}{J(S_1, S_2)}.$$

Let us remark that $J(S_1, S_2)$ depends on the spaces S_1 and S_2 and hence on the parameter h and so D depends on h also. Assume first that for all h

$$J(S_1(h), S_2(h)) > J_0 > 0, \quad (2.11)$$

and that for any $u \in H_1$

$$\inf_{\chi \in S_1(h)} \|u - \chi\|_{H_1} \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (2.12)$$

Then $u_{S_1(h)} \rightarrow u$ as $h \rightarrow 0$ in H_1 .

Let us now assume that

$$J(S_1, S_2) \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (2.13)$$

Then we have

THEOREM 2.3. [ABO]. *There exists $u_0 \in H_1$ such that*

$$\limsup \|u_0 - u_{S_1(h)}\|_{H_1} \neq 0 \quad \text{as } h \rightarrow 0. \quad (2.14)$$

We remark that also if (2.13) holds, for many (important in application) solutions u_0 we can get $u_{S_1(h)} \rightarrow u_0$ (also with optimal rate of convergence). For analysis of various aspects of Theorems 2.1–2.3 we refer to [BA1], [ABO].

For simplicity in this paper we will assume that $D_0 \geq D(H_1, H_2, S_1, S_2) > 0$ where D_0 is independent of $S_1(h), S_2(h)$. This occurs typically in self-adjoint elliptic coercive problems. Here we can use $H_1 = H_2 = E$, $\|u\|_E = |B(u, u)|^{1/2}$, $S_1 = S_2$ and obtain $D = 1$.

The finite element method is related to the selection of the family $S_1(h)$ and its approximation properties.

Let us now describe the h, p and $h - p$ version of the FEM. To explain the basic idea we assume that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $H_1 = H^1(\Omega) = H_2$ and

$$B(u, v) = \int_{\Omega} \left(\sum_{i=1,2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right) dx. \quad (2.15)$$

Then obviously (2.2), (2.3), (2.4) holds with $C = 1$ and $\gamma = 1$.

Consider a family \mathcal{T} of triangulations T_h . By $\tau \in T_h$ we denote the triangle (the element) of the triangulation T_h . As usual we will assume that two elements have either common side, common vertex or their intersection is empty. Further we will assume that minimal angle of the triangles is bounded below by θ_0 independently of the mesh of the family \mathcal{T} . (Instead of the minimal angle we can assume that the maximal angle is uniformly bounded from 2π [BA2]. This is used especially in 3 dimensions.) By $h_{\max}(T_h)$ resp. $h_{\min}(T_h)$ we denote the largest resp. smallest diameter of the triangles of the mesh T_h .

We will call the family \mathcal{T} the *quasiuniform* family if there exists $\alpha < \infty$ such that $h_{\max}(T_h)/h_{\min}(T_h) \leq \alpha$ for any $T_h \in \mathcal{T}$.

Let us now define the finite element spaces: $S(T_h, p) = \{u \in H^1(\Omega) : u|_{\tau \in T_h} = \text{polynomial of degree } p, p \geq 1\}$. (For simplicity we assume that p is the same for all $\tau \in T_h$.) By $N(T_h, p)$ we denote the dimension of $S(T_h, p)$, called number of degrees of freedom. Using now $S_1 = S_2 = S(T_h, p)$ we have $J(S_1, S_2) = 1$ in our model problem. To find $u_{S(T_h, p)}$ we have to solve system of $N(T_h, p)$ linear equations.

Let us now fix the degree $p = p_0 \geq 1$ and consider the meshes T_h with $h_{\max}(T_h) = h \rightarrow 0$. Then it can be readily seen that for any $u \in H^1(\Omega)$

$$\inf_{\chi \in S(T_h, p_0)} \|u - \chi\|_{H^1(\Omega)} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Hence $u_{S(T_h, p_0)} \rightarrow u$ as $h \rightarrow 0$ in the H^1 norm, i.e., a good accuracy of the finite element solution can be obtained by using sufficiently fine mesh. This approach is called the *h-version of the finite element method*.

Let us now fix the mesh T_{h_0} and consider the spaces $S(T_{h_0}, p)$, $p \rightarrow \infty$. Once more it is possible to show (see [BSK]) that

$$\inf_{\chi \in S(T_{h_0}, p)} \|u - \chi\|_{H^1(\Omega)} \rightarrow 0 \quad \text{as } p \rightarrow \infty$$

for any $u \in H^1(\Omega)$. Hence the approach to obtain finite element solution with desired accuracy by using sufficiently high p is called *the p -version of FEM*.

Combining simultaneously the mesh refining and increase of the degree p of the elements we obtain a version of FEM which is called *the $h-p$ version of FEM*.

The properties and the performances of these three versions depend on many factors as the concrete selection (generation) of the mesh, (a priori or adaptive), computer implementation, etc. We will here describe the problem of the approximation, i.e., of the error $\|u - u_S\|_{H^1(\Omega)}$ for these three versions.

3. The approximation problem

We have seen that (under our assumption)

$$\|u_S - u\|_{H^1(\Omega)} \leq D_0 \inf_{\chi \in S} \|u - \chi\|_{H^1(\Omega)},$$

where u is the exact solution of the problem. Coming back to (2.15) we consider the problem

$$B(u, v) = F(v) = \int_{\Omega} f v \, dx. \quad (3.1)$$

Its solution u is the solution of the Neumann problem.

$$(3.2a) \quad -\Delta u + u = f \quad \text{on } \Omega$$

$$(3.2b) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \quad (3.2)$$

The optimal selection of the finite element version resp. of the space S depends on the character of the solutions u of the class of solutions under consideration. Hence we have to characterize the set of functions S which are solutions of the class of problems we are interested in.

Let us first assume that Ω is convex and that we know only that $f \in L_2(\Omega)$. Then $u \in H^2(\Omega)$ and nothing more in general can be said about this solution. Hence we define

$$\mathcal{S}_1 = \{u \in H^1(\Omega) \mid u \in H^2(\Omega) \text{ with } \|u\|_{H^2} \leq A\}. \quad (3.3)$$

Assume now that Ω is a general polygon and A_i , $i = 1, \dots, m$, are the vertices of Ω with internal angles γ_i . Then the solution u of the problem (3.2) can be written in the form

$$u = u_1 + \sum_{l=1}^m c_l \varphi_l, \quad (3.4)$$

where $u_1 \in H^2(\Omega)$ and $\varphi_i = \kappa_i r^{\alpha_i} \cos \alpha_i \theta_i$ with (r_i, θ_i) being polar coordinates with the center in A_i , κ_i a cut-off C^∞ function and $\alpha_i = \frac{\pi}{\gamma_i}$. Hence we define the set \mathcal{S}_2 of the solution of interest

$$\mathcal{S}_2 = \left\{ u \in H^1(\Omega) \mid u = u_1 + \sum_{i=1}^m c_i \varphi_i, \quad \|u_1\|_{H^2} \leq A, \quad \sum_{\ell=1}^m c_\ell^2 \leq B \right\} \quad (3.5)$$

Let us now describe the set \mathcal{S}_3 which is appropriate for the case when f is analytic on $\bar{\Omega}$ (Ω is as before a polygon). This type of problem is the most relevant in practice.

Once more A_i , $i = 1, \dots, m$, be the vertices of the domain. Define

$$\beta = (\beta_1, \dots, \beta_m), \quad 0 < \beta_i < 1,$$

$$\Phi_{\beta+k}(x) = \prod_{i=1}^m r_i^{k+\beta_i}(x), \quad k \geq 0 \text{ integer}$$

with $r_i(x) = \text{dist}(x, A_i)$.

Further define for $\ell \geq 1$ the space $H_\beta^{k,\ell}(\Omega)$ with

$$\|u\|_{H_\beta^{k,\ell}(\Omega)}^2 = \|u\|_{H^{\ell-1}(\Omega)}^2 + \sum_{\ell \leq |\alpha| \leq k} \|\Phi_{\beta+|\alpha|-\ell} D^\alpha u\|_{L_2(\Omega)}^2,$$

$$D^\alpha = D^{(\alpha_1, \alpha_2)} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}},$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ are integers, $|\alpha| = \alpha_1 + \alpha_2$ and set

$$\mathcal{S}_3 = \mathcal{B}_\beta^\ell(\Omega) = \left\{ u \mid u \in H_\beta^{k,\ell}(\Omega), \quad k \geq \ell, \right. \\ \left. \|\Phi_{\beta+|\alpha|-\ell} D^\alpha u\|_{L_2(\Omega)} \leq C d^{k-\ell} (k-\ell)! \right\} \quad (3.6)$$

with $d \geq 1$ and $\ell = 2$.

The set \mathcal{S}_3 is a countably normed space.

We have

THEOREM 3.1. [BG3] [BG4] [BGO] *Let f be analytic on $\bar{\Omega}$. Then $u \in \mathcal{S}_3$ with some $\beta_i > 0$ depending on internal angles γ_i . (Analogous result holds for the elasticity problem (see [GB2]).)*

Let us now consider various families of finite element spaces of the h, p and $h-p$ version. Let N be the number of degrees of freedom. Then we define

$$R(N(T_{h,p}), \mathcal{S}) = \inf_{S(T_{h,p})} \sup_{u \in \mathcal{S}(\Omega)} \inf_{\chi \in S(T_{h,p})} \|u - \chi\|_{H^1}, \quad (3.7)$$

where $S(T_{h,p})$ can be family of spaces based on the h -version: $S(T_h, p_0)$, the p -version: $S(T_{h_0}, p)$ or the $h-p$ version: $S(T_h, p)$.

We are interested in the performance of the h, p and $h-p$ version if the solution of the problem belongs to the set \mathcal{S}_i , $i = 1, 2, 3$.

First we consider the set \mathcal{S}_1 .

THEOREM 3.2. We have

a) The h -version:

$$C_1(p_0)(N(T_h, p_0))^{-1/2} \leq R(N(T_h, p_0), \mathcal{S}_1) \leq C_2(p_0)(N(T_h, p_0))^{-1/2} \quad (3.8)$$

b) the p -version:

$$C_1(T_h)(N(T_{h_0}, p))^{-1/2} \leq R(N(T_{h_0}, p), \mathcal{S}_1) \leq C_2(T_h)(N(T_{h_0}, p))^{-1/2} \quad (3.9)$$

c) the $h-p$ version:

$$C_1(N(T_h, p))^{-1/2} \leq R(N(T_h, p), \mathcal{S}_1) \leq C_2(N(T_h, p))^{-1/2}. \quad (3.10)$$

In addition the rate $N^{-1/2}$ is achieved for the h -version for the family of quasi-uniform meshes and $p_0 = 1$. The proof utilizes theory of the n -width (see [P]) and a priori estimates. (For the p -version see [BSK].)

Theorem 3.2 shows that the three versions are equivalent with respect to the number of degrees of freedom N . It does not mean that these three versions are equivalent for the numerical computations. For example the system of linear equations for the p -version is less sparse than for the h -version which influences the CPU time.

Let us now consider the set \mathcal{S}_2 and let $\gamma_0 = \max(\gamma_i)$, where γ_i is the internal angle in the vertex A_i .

THEOREM 3.3. We have for

a₁) the h version with quasiuniform meshes:

$$\begin{aligned} C_1(p_0)(N(T_h, p_0))^{-1/2 \min(1, \pi/\gamma_0)} &\leq \\ &\leq R(N(T_h, p_0), \mathcal{S}_2) \leq C_2(p_0)(N(T_h, p_0))^{-1/2 \min(1, \pi/\gamma_0)}, \end{aligned} \quad (3.11)$$

a₂) the h version with arbitrary (i.e. properly refined) meshes:

$$C_1(p_0)N(T_h, p_0)^{-1/2} \leq R(N(T_h, p_0), \mathcal{S}_2) \leq C_2(p_0)(N(T_h, p_0))^{-1/2}, \quad (3.12)$$

b) the p -version

$$R(N(T_h, p_0), \mathcal{S}_2) \leq C_2(T_{h_0})(N(T_{h_0}, p))^{-\min(1/2, \pi/\gamma_0)}, \quad (3.13)$$

c) the $h-p$ version

$$C_1(N(T_h, p))^{-1/2} \leq R(N(T_h, p), \mathcal{S}_2) C_2(N(T_h, p))^{-1/2}. \quad (3.14)$$

For the proof of upper estimate for the p -version see [BSK] [BS1].

R e m a r k. In [BSK], [BS1] and [BS2] a more general statement is proven. In [BSK] slightly weaker result was proven, namely instead $\min(1/2, \pi/\gamma_0)$ only $\min(1/2, \pi/\gamma_0 - \varepsilon)$, $\varepsilon > 0$ arbitrary with $C_2(T_{h_0}, \varepsilon)$ was obtained. The term ε was removed in [BS1].

We see that for the h -version with a uniform mesh the singularity of the solution in the neighborhood of a corner is governing the accuracy. For the properly refined mesh the error is as when no corner is present. The p -version is able to “absorb” better the singular behavior than the h -version with the quasiuniform mesh. The $h - p$ version gives the rate of convergence as if the singular behavior in the corners would not be present.

Finally let us address the set \mathcal{S}_3

THEOREM 3.4. *We have for*

a) *the h version with arbitrary (i.e. with) properly refined mesh*

$$\begin{aligned} C_1(p_0)(N(T_h, p_0))^{-p_0/2} &\leq R(N(T_h, p_0), \mathcal{S}_3) \leq \\ &\leq C_2(p_0)(N(T_h, p_0))^{-p_0/2}, \end{aligned} \quad (3.15)$$

b) *the $h - p$ version [BG1] [BG2]*

$$R(N(T_h, p), \mathcal{S}_3) \leq C \exp\left(-\mu \sqrt[3]{N(T_h, p)}\right). \quad (3.16)$$

The exact value $\mu > 0$ in (3.16) is not explicitly known. For analog of this constant in one dimension we refer to [GB1]. The lower bound in (3.16) is not known. For a complete analysis in one dimension we refer to [GB1].

Theorem 3.4 shows that in the case when the input data are analytic, as it is in most cases in practice, the exponential rate can be achieved.

We note that the mentioned theorems can be generalized.

R e m a r k. When the solution u_0 is analytic on $\bar{\Omega}$ then for the p -version we have $R < C \exp(-\mu \sqrt[2]{N})$. We addressed here only the result for the Problem 3.2. Similar results hold also for the equation of elasticity.

We discussed only 2 dimensional problem. In three dimensions the solution is more complex. The analog of the space \mathcal{B}_β^2 is more complicated. The solution has singularities along the edges and in the neighborhoods of the vertices. In contrast to the two dimensional definition of \mathcal{B}_β^ℓ which is based on weighted isotropic spaces, in 3 dimensions we have to utilize weighted anisotropic spaces because the solution is smooth along the edges but singular in the perpendicular direction to the edges. Still more complicated behavior is in the vertices where the edges are joining together. For the scalar second order problem in 3 dimensions the spaces $\mathcal{B}_\beta^\ell(\Omega)$ were analyzed in [GB3] and analog to the Theorem 3.1 was

proven. The upper bound for $R(N(T_{h,p}))$ is also analyzed and in contrast to the two dimensional case we get here [BG6]

$$R \leq C \exp(-\mu \sqrt[5]{N}) \quad (3.17)$$

The p -version in 3 dimensions is analyzed in [D1] [D2].

In practical computations the meshes are created by a mesh generator using experience of the user. In the h -version the mesh is usually not designed optimally and hence the rate usually is not optimal. In the $h-p$ version in practice often a fixed properly refined mesh is used and only the degree p is increased. Hence we, in the strict sense, deal with the p -version; nevertheless in a practical range of accuracy the method behaves approximately as the $h-p$ version.

The meshes can be selected a priori as we mentioned or in an adaptive mode. In the h -version the degree is fixed and mesh is adaptively constructed. In the p -version the degrees of elements are adaptively chosen (non uniformly). In the $h-p$ version the mesh and degrees are selected simultaneously.

Major importance plays here the a-posteriori error estimation which is also basis for the adaptive procedure. The a-posteriori error estimation for the h -version is addressed in [BSU1] [BSU2] and in references there. The a-posteriori error estimation for the p -version is usually based on the extrapolation procedure, see, e.g., [SzB].

The $h-p$ version properly implemented is very effective one which can be seen on the recent successful development and practical use of the commercial codes based on the p and $h-p$ versions. Nevertheless we will not here discuss the implementational and engineering aspects of the $h-p$ version.

For additional references see [BG5] [BS4].

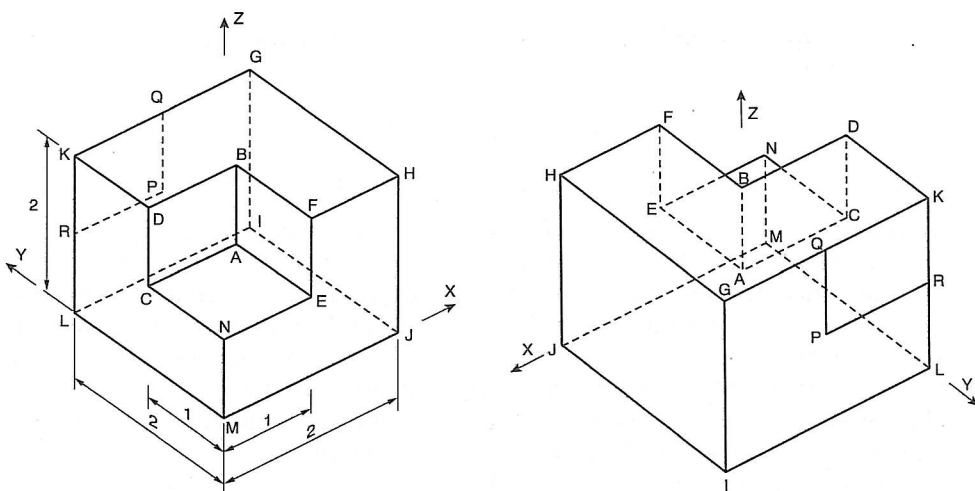
4. A Numerical Example

As numerical example we will discuss the three dimensional problem for the Laplace equation on the domain Ω shown in Fig. 4.1, where the vertices and dimensions are depicted.

The used boundary conditions are:

- i) $u = 0$ on the faces $A-C-E-N$, $A-B-EF$, $A-B-C-D$, $B-F-H-G-Q-K-D$, $G-H-I-J$, $K-Q-P-R$
- ii) $\frac{\partial u}{\partial n} = 0$ on the faces $D-K-R-L-M-N-CN-M-J-H-F-E$, $Q-G-I-L-R-P$
- iii) $\frac{\partial u}{\partial v} = \cos \frac{\pi x}{4}$ on the face $I-J-L-M$.

The solution has *edge singularities* along the edges $A-B$, $A-E$, $A-C$, $B-D$, $B-F$, $G-H$, $Q-K$, $Q-P$, $P-R$, $I-J$ and the *vertex singularities*


 FIG. 4.1. Domain Ω

are located in the neighborhood of the vertices A, B, Q, P . There are no vertex singularities in other ends of the singular edges where only edge singularities are present.

We did use 7 types of meshes $M_1 - M_7$ which differ by the strength of the refinement along the edges and vertices. On every mesh we used elements of increasing degree p . The meshes were constructed so that with proper relations between the type of meshes and degrees of elements, we should see (by the theoretical results) the rate of convergence (at least) $\exp(-\gamma\sqrt[5]{N})$. In the Fig. 4.2 we show the relative error in the log scale as function of the number of degrees of freedom N depicted in the scale $N^{1/5}$. Then the function $\exp(-\gamma\sqrt[5]{N})$ is a straight line. We see in Fig. 4.2 that the proper selection of the degrees p depending on M_i leads to the exponential convergence rate $\exp(-\gamma N^\alpha)$ where $\alpha = 1/5$ (in fact is closer to $1/4.5$). The energy of the exact solution was computed by extrapolation and the squared energy norm of the error was completed as the difference between the exact and finite element energy.

The example was computed by Dr. B. A n d e r s o n (Aeronautical Research Institute of Sweden) using the program STRIPE. The author would like to express his gratitude for these computations.

5. Various additional features of the p and $h - p$ version of the FE method

We have addressed in the previous sections only the Neumann boundary condition. In the case of nonhomogeneous Dirichlet boundary condition, the boundary condition has to be replaced by a function which is in the trace space

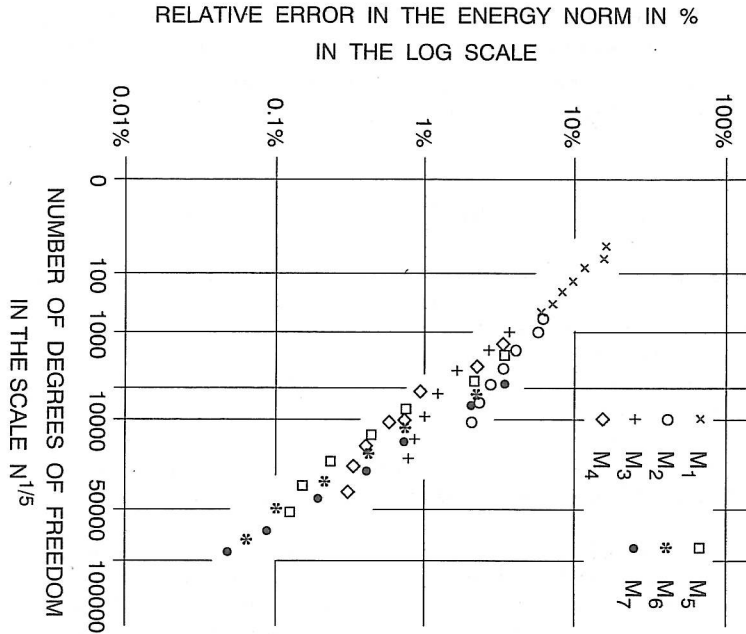


FIG. 4.2. The error of the FE solution measured in the energy norm

of S . For an analysis of this question we refer to [BGS] [BS3].

We treated the singular behavior of the solution in the neighborhood of the corners by a proper refinement of the mesh. In the p version there is another possibility which is very effective in various circumstances. Here we use special mappings related to the p -version which leads to high rate of convergence. This method can be easily implemented [BO1], [BO2].

We have addressed the p and $h-p$ version for the equations of second order. The problem of equations of higher order is addressed in [G1], [S].

As we said earlier, the finite element method reduces the problem to a system of linear equations. This system is routinely solved by the direct method typically up to the size of 5,000–20,000. For larger systems iterative method is used using specific properties related to the p version. We refer to [BC], [BE], [BGP] and [MA]. The p and $h-p$ version is widely used today in engineering. For some engineering results we refer to [AB], [Sz].

The p and $h-p$ version avoids often the problem of “locking” in the case of nearly degenerated problems as in the case of nearly incompressible materials

[V], [J] or shells [Pi]. The $h-p$ version is also naturally related to the hierarchic modelling of plates (see, e.g., [BL]).

So far we addressed the error measured in the energy norm. For the study of the error of the p -version measured in the L_2 norm we refer to [JS].

The p and $h-p$ version is very naturally applicable for other methods typically for the boundary element method [PS].

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