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HARDY SPACE METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Hardy space methods have lead to remarkable progress on nonlinear partial differential equations with critical growth. The results obtained by a variety of authors include the regularity theory for weakly harmonic maps, existence results for the two-dimensional instationary Euler equations with vortex sheet initial data and the Lipschitz parametrization of $W^{2,2}$ surfaces. This paper gives a quick review of the basic tools needed and discusses their application.

1. Introduction

In recent years research by a variety of authors lead to remarkable progress in the study of nonlinear partial differential equations with critical growth through the use of Hardy space methods. The idea is simple. A typical difficulty when dealing with such equations is that the nonlinear term is a priori only known to be in L^1 while there is no good elliptic theory in L^1. There is, however, a well-established regularity theory in the slightly smaller Hardy space $\mathcal{H}^1$ and Coifman, Meyer, Lions and Semmes [CLMS] recently discovered that certain nonlinear quantities which at first glance only seem to be in L^1 are in fact in $\mathcal{H}^1$. In the following I will briefly discuss applications of this idea to the (longstanding) regularity problem for harmonic maps, good parametrizations of surfaces and the two-dimensional Euler equations. Further applications can be found in [CLMS]. First, it is time to recall the definition of $\mathcal{H}^1$ and some of its properties.

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2. The Hardy space $\mathcal{H}^1$

The *Hardy spaces* $\mathcal{H}^p$ (with $0 < p < \infty$) have been an object of interest in complex analysis for a long time. A breakthrough came in the early 70's when it was discovered that they possess a simple real variable characterization and that various seemingly unrelated definitions are equivalent. A classic reference is the article by Fefferman and Stein [FS] (see also [Se] for a short introduction to Hardy spaces). In the following we only consider $\mathcal{H}^1(\mathbb{R}^n)$ which can be defined as follows.

Let ϕ we a smooth function with compact support in the unit ball and suppose that $\int \phi = 1$. For a function $f \in L^1(\mathbb{R}^n)$ define the regularized maximal function by

$$f^*(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n} \varepsilon^{-n} \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)\, dy \right| . \tag{1}$$

DEFINITION 1. The function f is in $\mathcal{H}^1(\mathbb{R}^n)$ if and only if $f^* \in L^1(\mathbb{R}^n)$.

The expression $\|f^*\|_{L^1}$ defines a norm on $\mathcal{H}^1$ and different choices of ϕ lead to equivalent norms. Note that f^* is closely related to the Hardy–Littlewood maximal function but that it takes into account possible cancellations since the absolute value is taken outside the integral. By the Lebesgue point theorem $\mathcal{H}^1$ is a subspace of L^1. Note also that f can only be in $\mathcal{H}^1$ if $\int f = 0$ since otherwise f^* only decays like $|x|^{-n}$ at ∞.

A crucial property of $\mathcal{H}^1$ is that elliptic operators (or more generally singular integral operators) behave well on that space while the same is not true in L^1. We will only need the following result.

THEOREM 2. ([FS]). *Let $f \in \mathcal{H}^1$ and let u be a solution of*

$$-\Delta u = f \quad \text{in} \quad \mathbb{R}^n .$$

Then u can be written in the form $u = u_0 + H$ where H is harmonic and u_0 satisfies

$$\|\nabla^2 u_0\|_{\mathcal{H}^1} \le C \|f\|_{\mathcal{H}^1} . \tag{2}$$

If $n = 2$ then one can use the embeddings (see [Ad], Lemma 5.8) $W^{2,1} \hookrightarrow C^0$ and $W^{2,1} \hookrightarrow W^{1,2}$ to show that $u_0 \in W^{1,2} \cap C^0$ and

$$\|\nabla u_0\|_{L^2} + \|u_0\|_{L^\infty} \le \|f\|_{\mathcal{H}^1} . \tag{3}$$

The space $\mathcal{H}^1$ has various other interesting properties. In particular its dual is the John–Nirenberg space BMO of functions of bounded mean oscillation, see [FS], [Se], [Tor] ... for many further details.

In the last section we will also need local versions of the Hardy space. To this end let

$$f^{**}(x) = \sup_{1>\varepsilon>0} \left| \int_{\mathbb{R}^n} \varepsilon^{-n} \phi\Big(\frac{x-y}{\varepsilon}\Big) f(y)\, dy \right| .$$

The local Hardy space h^1 introduced by Goldberg [Go] consists of all functions f for which $\|f^{**}\|_{L^1}$ is finite and this quantity gives a norm on h^1. Finally $\mathcal{H}^1_{\mathrm{loc}}$ consists of all functions f for which $f^{**} \in L^1_{\mathrm{loc}}$ (see also [Zhe] and [EM], Section 5).

3. Some nonlinearities are special

It has been known for some time now that certain nonlinear expressions such as the Jacobien $\det \nabla u$ of a map $u : \mathbb{R}^n \to \mathbb{R}^n$ behave particular well under weak convergence and this observation has been crucial in such diverse fields as nonlinear elasticity and quasiregular maps (see [Mo], [Re], [Ba]).

More recently it emerged that the Jacobien also enjoys special integrability properties. The following result was shown in [Mu1], [Mu2] (inspired by results of Zhang [Zh]): if $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and if $\det \nabla u \geq 0$ a.e. then $\det \nabla u \log(2 + \det \nabla u)$ is locally integrable (see [IS], [BFS], [IL], [Mi] for further developments). The space $L \log L$ is well known in harmonic analysis and Coifman, Lions, Meyer and Semmes, partially motivated by the above results, established the crucial connection to $\mathcal{H}^1$.

THEOREM 3. ([CLMS]) *If $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ then $\det \nabla u \in \mathcal{H}^1$ and*

$$\|\det \nabla u\|_{\mathcal{H}^1} \leq C \|\nabla u\|^n_{L^n} . \tag{4}$$

The connection between weakly continous quantities and quantities that enjoy higher integrability does not seem to be coincidental. See [CLMS] for results similar to the one above for the quantities that appear in the theory of compensated compactness which was developed by Murat and Tartar (see [Ta1]). Combining Theorems 2 and 3 one can recover the remarkable earlier results of Wente [We] (see also Brezis and Coron [BC] and Tartar [Ta2], [Ta3], [Ta4]) on the equation $-\Delta u = \det \nabla v$ (Wente considered a slightly more special case but the same ideas apply).

A very short proof of the theorem which starts from the definition of $\mathcal{H}^1$ given in Section 2 and uses only the fact that the Jacobien can be written as a divergence (see [Mo], [Re], [Ba]), the Sobolev–Poincare inequality and L^p $(p > 1)$ estimates for the Hardy–Littlewood maximal function (see [St]) appears in [Li] (see also [Ev]).

4. Weakly harmonic maps

Let B be the unit ball in $\mathbb{R}^2$ and let $u \in W^{1,2}(B, \mathbb{R}^{n+1})$. Suppose that

$$u(x) \in S^n \quad \text{a.e.}$$

We say that u is weakly harmonic if

$$-\Delta u = u|\nabla u|^2 \tag{5}$$

in the weak sense, i.e, if u is a stationary point of the Dirichlet integral subject to the constraint $u \in S^n$. It had been open for a long time whether all such maps are smooth although that was known under additional assumptions (see, e.g., the references in [He1], [He2], [Ri]).

THEOREM 4. ([He1]) *Weakly harmonic maps are smooth.*

Hélein also showed that the regularity theorem still holds if the target S^n is replaced by an arbitrary Riemannian manifold ([He2]). His results where generalized by Evans [Ev] and Bethuel [Be] who proved partial regularity of weakly harmonic maps from higher dimensional domains provided that a monotonicity formula holds. Rivière [Ri] gave counterexamples to regularity if that condition is not imposed.

A sketch of the proof. The first observation is that equation (5) can be written in divergence form. This is essentially a consequence of the symmetries of S^n and Noether's theorem and to the author's knowledge was first observed in the derivation of the Eriksen–Leslie theory of liquid crystals (see [Le]), see also Shatah ([Sh]). Specifically a map $u\colon B \to S^n$ (in $W^{1,2}$) is a weak solution of (5) if and only if

$$\operatorname{div} b^{ij} = 0, \quad \text{for all} \quad i, j = 1, \ldots, n+1,$$

where

$$b^{ij} = u^i \nabla u^j - u^j \nabla u^i.$$

In particular there exist functions w^{ij} such that $b^{ij} = \nabla^\perp w^{ij}$, where $\nabla^\perp = (-\partial/\partial x^2, \partial/\partial x^1)$. A short calculation using $|u| = 1$ shows that

$$u^i |\nabla u|^2 = \sum_j \det(\nabla w^{ij}, \nabla u^j)$$

and (after a suitable localization argument) it follows from Theorems 2 and 3 that u is continuous. Standard results for harmonic maps then imply that u is smooth. □

It should be noted Helein's original proof does not use $\mathcal{H}^1$, in fact the result in the appendix of [BC] gives slightly more precise results. The proof for general targets, as well as the partial regularity results do, however, make use of $\mathcal{H}^1$ (see also the references in [Ev] and [Be]).

5. Lipschitz parametrization of $W^{2,2}$ graphs

Let $w \in W^{2,2}(\mathbb{R}^2)$ and let $\Gamma \subset \mathbb{R}^3$ be the graph of w. Although $W^{2,2}$ functions need not to be Lipschitz, Toro established the striking fact that Γ has a bilipschitz parametrization.

THEOREM 5. ([To]) *There exists a parametrization $f : \mathbb{R}^2 \to \Gamma$ and a constant C (depending only on $\|\nabla^2 w\|_{L^2}$) such that for all $x, y \in \mathbb{R}^2$*

$$C^{-1}|x-y| \le \big|f(x) - f(y)\big| \le C|x-y| \, .$$

Toro's proof proceeds by an explicit iterative construction. Here I would like to sketch the alternative proof of [MS].

Sketch of proof. For convenience we assume that w is smooth with compact support and show that the Lipschitz constant C only depends on $\|\nabla^2 w\|_{L^2}$. Let f be a conformal parametrization of Γ, i.e., $f_{x_1} \cdot f_{x_2} = 0$ and $|f_{x_1}| = |f_{x_2}| = e^u$. Since w has compact support u is harmonic outside a large ball and hence has a limit at ∞. Replacing $f(x)$ by $f(\lambda x)$ if necessary we may assume that this limit is zero.

The Gauss curvature K of Γ satisfies

$$-\Delta u = e^{2u}(K \circ f) \, .$$

Let $N \colon \Gamma \to S^2$ be the Gauss map that associates to each point $p \in \Gamma$ its normal $N(p)$ (given by $f_{x_1} \wedge f_{x_2} / |f_{x_1} \wedge f_{x_2}|$) and let $\phi = N \circ f$. Since K is the Jacobien of the Gauss map we have

$$e^{2u}(K \circ f) = \det \nabla \phi \, .$$

By the conformal invariance of the Dirichlet integral and a short calculation

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx = \int_{\Gamma} |\nabla N|^2 \le \int_{\mathbb{R}^2} |\nabla^2 w|^2 \, dx \, .$$

It now follows from Theorems 2 and 3 (and (4)) that

$$\sup |u| \le C \int_{\mathbb{R}^2} |\nabla^2 w|^2 \, dx \, . \tag{6}$$

Here we used the fact that bounded harmonic functions are constant and that u approaches zero at infinity. In applying Theorem 3 we glossed over the fact that ϕ takes its values in S^2 rather than $\mathbb{R}^2$. This causes no problem since Γ is a graph and therefore ϕ only takes its values in the upper half-sphere which can be mapped to a ball in $\mathbb{R}^2$ by a volume-preserving diffeomorphism.

From (6) we obtain the upper bound in the theorem. For the lower bound a short argument is needed to compare the extrinsic and the intrinsic distance on Γ (see [MS], Section 5). □

Similar techniques can be applied to (smooth) immersed surfaces $M \hookrightarrow \mathbb{R}^n$ which need not be graphs. The fundamental assumption in this case is that the second fundamental form A satisfies

$$\int_M |A|^2 < \infty . \tag{7}$$

By a result of Huber [Hu] such surfaces admit a conformal parametrization $f : S \setminus \{a_1, \dots, a_q\} \to M \hookrightarrow \mathbb{R}^n$ where S is a compact Riemann surface.

THEOREM 6. ([MS]) *Let M be as above and suppose that M is complete, connected and non-compact. Then the immersion f is proper, i.e, $f(x) \to \infty$ if $x \to a_i$. If, in addition, $\int_M |A|^2 < 4\pi$, then the conformal type of M is C and $f : C \to M \hookrightarrow \mathbb{R}^n$ is an embedding.*

6. Instationary Euler equations

We are interested in solutions of the two-dimensional instationary Euler equations

$$u_t + \operatorname{div} u \otimes u = -\nabla p , \tag{8}$$

$$\operatorname{div} u = 0 , \tag{9}$$

with "rough" initial data. More precisely we suppose that $u_0(x) := u(0,x)$ satisfies

$$\omega_0 := \operatorname{curl} u_0 \in \mathcal{M} ,$$

where $\mathcal{M}$ denotes the space of Radon measures on $\mathbb{R}^2$. The condition $\operatorname{div} u_0 = 0$ and ω_0 determine u_0 up to a gradient of a harmonic function. Choose a normalization by requiring that u_0 can be written as $u_0(x) = \lambda(|x|)x^{\perp} + u_1(x)$ where λ is smooth and where $u_1 \in L^2(\mathbb{R}^2; \mathbb{R}^2)$.

The case where ω_0 is a one-dimensional measure concentrated on a curve corresponds to so-called vortex-sheet initial data which are believed to be of great practical importance and have been studied extensively numerically (see, e.g., [Ma1] and the references therein).

A standard procedure to obtain solutions for such rough initial data u_0 is to approximate them by smooth data $u_{0\varepsilon}$ and to pass to the limit $\varepsilon \to 0$ in the corresponding solutions u_ε. By the usual estimates (see [Yu], [Ka], [DM], [De]) one obtains bounds (independent of ε) for $u_\varepsilon(t,\cdot) \in L^2_{\mathrm{loc}}$, $\omega_\varepsilon(t,\cdot) \in L^1_{\mathrm{loc}}$, uniformly for $t \in [0,T]$ and there exists a sequence $\varepsilon_j \to 0$ such that

$$\begin{aligned} u_{\varepsilon_j} &\rightharpoonup \bar{u} \quad \text{weakly in} \quad L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^2) \\ u_{\varepsilon_j} &\to \bar{u} \quad \text{a.e. in} \quad [0,T] \times \mathbb{R}^2 . \end{aligned}$$

If instead of weak convergence in L^2_{loc} one had strong convergence then one could easily show that $\bar{u}$ is a solution of (8) and (9). Such strong convergence may, however not hold in general (cf. [Ma1]). Di Perna and Majda (see [DM], where further references can be found) begun a detailed investigation into the question whether a limit of (exact or approximate) solutions of Euler's equation is again a solution if no strong convergence holds. In particular they showed that this is the case for the stationary Euler equation (in two dimensions) by a careful analysis of the "concentration set" where strong convergence fails. It has so far not been possible to extend that approach to the time-dependent case, mainly due to poor control in time (but see [Lo], [Bi]).

Delort [De] took a different approach. First note that u is a weak solution of (8), (9) if and only if

$$\omega_t = \left((u^2)^2 - (u^1)^2\right)_{x_1 x_2} + (u^1 u^2)_{x_2 x_2} - (u^1 u^2)_{x_1 x_1} , \tag{10}$$

where $\omega = \operatorname{curl} u$ or

$$u = K * \omega , \qquad K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2} . \tag{11}$$

Delort then showed the following, using the representation (11) of u.

Theorem 7. *If $\omega_{0\varepsilon} \geq 0$ then one has with the above notation*

$$u^1_{\varepsilon_j} u^2_{\varepsilon_j} \rightharpoonup \bar{u}^1 \bar{u}^2, \quad (u^1_{\varepsilon_j})^2 - (u^2_{\varepsilon_j})^2 \rightharpoonup (\bar{u}^1)^2 - (\bar{u}^2)^2$$

in the sense of distributions. In particular $\bar{u}$ is a solution of (8), (9).

For variants and generalizations of Delort's argument see [Ge] and [Ma2].

In [EM] (see also [Se]) an attempt was made to understand Delort's result in the context of Hardy spaces. The main observation is the following.

Theorem 8. *Let $\psi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and suppose that $\Delta\psi \geq 0$. Then $\psi_{x_1}\psi_{x_2}$ and $(\psi_{x_1})^2 - (\psi_{x_2})^2$ are in $\mathcal{H}^1_{\mathrm{loc}}$ and for each $\phi \in C_0^\infty(\mathbb{R}^2)$ one has*

$$\|\phi\psi_{x_1}\psi_{x_2}\|_{h^1} + \|\phi((\psi_{x_1})^2 - (\psi_{x_2})^2)\|_{h^1} \leq C \int_{B(0,R)} |D\psi|^2\, dx\,,$$

where C and R only depend on ϕ (but not on ψ).

The proof is similar in spirit to Delort's and relies on the definition of $\mathcal{H}^1_{\mathrm{loc}}$ given above and the representation (11) of u.

For the application to the Euler equation note that due to the equation $\operatorname{div} u = 0$, u can be expressed as $u = \nabla^\perp\psi$ so that $\omega = \operatorname{curl} u = \Delta\psi$.

Delort's result can then be recovered from the following fact (see [EM] for details of the argument).

Theorem 9. ([JJ], [EM]). *Suppose that $f_j \to f$ a.e. and that $\|f_j\|_{h^1}$ is bounded. Then $f_j \rightharpoonup f$ in the sense of distributions.*

Note that the result fails if h^1 is replaced by L^1. Using the Hardy space approach one also obtains a new estimate for the streamfunction ψ, namely $\psi_t \in L^\infty([0,T], L^1_{\mathrm{loc}})$ or more precisely

$$\|\phi\psi_t(t,\cdot)\|_{h^1} \leq C(\phi, T)\,, \qquad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^2), \quad t \in [0,T]\,.$$

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