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THE YANG–MILLS GRADIENT FLOW IN FOUR DIMENSIONS

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ABSTRACT. This paper summarizes the results of [6], as presented at EQUADIFF 8, 1993.

Let $\pi\colon \eta \to (M,g)$ be a vector bundle over a compact m-dimensional Riemannian manifold with fibre $\pi^{-1}(x) \cong \mathbb{R}^n$ and structure group $G \subset \mathrm{SO}(n)$. We denote $\mathfrak{g} = T_{\mathrm{id}}G$ the Lie algebra of G.

Consider G-invariant connections ∇ on η and the induced operators

$$\Omega^0(\eta) \xrightarrow{D} \Omega^1(\eta) \to \dots$$
$$\Omega^0(\operatorname{ad}\eta) \xrightarrow{D} \Omega^1(\operatorname{ad}\eta) \to \dots$$

acting on differential forms on η or its associated bundles.

Given a reference connection, any such D can be expressed

$$D = D_{\mathrm{ref}} + A$$

in terms of a connection 1–form $A \in \Omega^1(\operatorname{ad}\eta)$. Locally, A is a $\mathfrak{g}$-valued differential 1–form.

The curvature of D is the 2–form

$$F(D) = D \circ D \in \Omega^2(\operatorname{ad}\eta)\,.$$

The Yang–Mills action of D is

$$\mathrm{YM}(D) = \frac{1}{2}\int_M |F(D)|^2\,dx\,,$$

where we define the L^2–norm using the base metric g and a G–invariant metric on the fibres of η.

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Using the expansion

$$F(D+a) = (D+a)\circ(D+a) = D\circ D + Da + a\wedge a = F(D) + Da + a\wedge a$$

we can compute the first variation

$$\frac{d}{d\varepsilon}\,\mathrm{YM}(D+\varepsilon a)\,\big|_{\varepsilon=0} = \big(F(D), Da\big) = \big(D^*F(D), a\big)\,,$$

where D^* is the Hodge-adjoint operator of D.

By definition, D is a Yang–Mills connection, if YM is stationary at D; that is, if

$$D^*F(D) = 0\,.$$

This is a degenerate elliptic system.

In order to be able to analyze the structure of the set of solutions to the Yang–Mills equation by Morse theory it is necessary to understand the L^2–gradient flow for the Yang–Mills functional

$$\frac{d}{dt}D = -D^*F(D) \tag{1}$$

with initial data

$$D(0) = D_0 = D_{\mathrm{ref}} + A_0\,. \tag{2}$$

Multiplying (1) by $\frac{d}{dt}D$, we obtain the energy inequality

$$\int_0^T\int_M \Big|\frac{d}{dt}D\Big|^2\,dx\,dt + \mathrm{YM}\big(D(T)\big) \le \mathrm{YM}(D_0) \tag{3}$$

for any smooth solution of (1), (2); in fact equality holds.

Our main result is the following local existence result for (1) in dimension $m = 4$ with initial data (2) of finite energy.

Theorem 1. *Let $m = 4$, $D_0 = D_{\mathrm{ref}} + A_0$, $A_0 \in H^{1,2}\big(\Omega^1(\mathrm{ad}\,\eta)\big)$. Then there exists $0 < T \le \infty$ and a weak solution $D(t) = D_{\mathrm{ref}} + A(t)$ to (1), (2) such that*

$$A \in C^0\Big([0,T[\,; H^{1,2}\big(\Omega^1(\mathrm{ad}\,\eta)\big)\Big)$$

and satisfying (3). T is maximal with the property that for all $T' < T$ there is $R > 0$ such that

$$\sup_{\substack{x_0\in M\\ 0\le t\le T'}} \int_{B_R(x_0)} \big|F(t)\big|^2 \le \varepsilon_0\,,$$

where $\varepsilon_0 = \varepsilon_0(\eta) > 0$ is independent of D. D is gauge-equivalent to a smooth solution $\bar{D}$ for $0 < t < T$. D is smooth, if D_0 is. D is (locally) unique if D (respectively, D_0) is irreducible in the sense of (7) below.

Remarks.

For $m = 2, 3$ Råde [3] obtained global existence for (1), (2) and exponential convergence of $D(t)$ to a Yang–Mills connection D_∞ as $t \to \infty$.

If $m > 4$, even smooth solutions may blow up in finite time ([2]).

In the limiting case $m = 4$, in case of a holomorphic vector bundle over a compact Kähler surface, Donaldson obtained global existence of solutions of (1), (2) for smooth data, and their asymptotic convergence if the underlying bundle is stable; see [1].

Major open problems are existence and uniqueness of suitable weak solutions in dimensions $m \geq 5$ and the possibility of finite-time blow-up in dimension $m = 4$. There is a rather strong analogy with the evolution problem for harmonic maps; see for instance [5] for a survey.

Proof. We sketch the main ideas.

i) The proof of the local existence part of the theorem — like Donaldson's proof — makes use of the gauge-invariance of the Yang–Mills functional and "De Turck's trick": If D is a solution of (1), (2) and if

$$\bar{D} = S^*(D) = S^{-1} \circ D \circ S = D_{\text{ref}} + \bar{A} \tag{4}$$

is the corresponding family of pull-back connections under gauge transformations $S(t)\colon \eta \to \eta$, then there holds

$$\frac{d}{dt}\bar{D} + \bar{D}^* F(\bar{D}) = \bar{D}s, \quad s = S^{-1}\frac{d}{dt}S\,. \tag{5}$$

Choosing

$$s = -\bar{D}^* \bar{A}\,, \tag{6}$$

equation (5) becomes a parabolic system for the unknown connection 1–form $\bar{A}$, which can be solved for small time $t > 0$. Finally, S — and hence D, via (4) — can be recovered from (5) and (6).

To overcome certain technical difficulties that may arise in the case of data which are not smooth we found it useful to work with a time-dependent background connection

$$D_{\text{bg}}(t) = D_{\text{ref}} + A_{\text{bg}}(t)\,,$$

where $A_{bg}(0) = A_0$ and $A_{\text{bg}}(t)$ is smooth for $t > 0$.

ii) Observe that uniqueness for the gauge-equivalent problem (5) requires $\ker \bar{D} \cap \Omega^0(\operatorname{ad}\eta) = \{0\}$. In the smooth case, this condition is equivalent to the condition that $\bar{D}$, respectively D, is irreducible in the sense that the isotropy subgroup

$$\Gamma = \{S\,; S^*(D) = D\}$$

is trivial. In the general case, we require strong irreducibility in the sense

$$\exists\, C > 0 \quad \forall s \in \Omega^0(\operatorname{ad}\eta)\colon \|Ds\|_{L^2} \geq C^{-1}\|s\|_{H^{1,2}}\,. \tag{7}$$

If (7) holds, by a global analogue of Uhlenbeck's theorem [8] we can achieve the following gauge condition

$$\bar{D} = D_{\mathrm{bg}} + \bar{A}, \quad \bar{D}^* \bar{A} = 0. \tag{8}$$

In view of

$$F(\bar{D}) = F(D_{\mathrm{bg}}) + \bar{D}\bar{A} - \bar{A} \wedge \bar{A},$$

equations (5), (8) then take the form

$$\frac{d}{dt}\bar{A} + \bar{\Delta} + \cdots = \bar{D}s,$$

where $\bar{\Delta} = \bar{D}^*\bar{D} + \bar{D}\bar{D}^*$ is the Hodge–Laplacian. Using again (7), the uniqueness of $\bar{A}$, that is, $\bar{D}$ and hence the uniqueness of D follow. □

Remark the similarity between the gauge-normalized problem (5), (8) and the Navier–Stokes system for the velocity field v and pressure p of a viscous fluid

$$\frac{d}{dt}v - \operatorname{div}(\operatorname{grad} v - v \otimes v) = -\operatorname{grad} p,$$
$$\operatorname{div} v = 0.$$

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