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EQUIVALENCE OF DIFFERENTIAL EQUATIONS AND DIFFERENTIAL ALGEBRAS

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ABSTRACT. In a space of nonlinear ordinary differential equations we consider two equivalence relations. Static equivalence means equivalence up to nonlinear transformations of dependent variables. A much weaker dynamic equivalence uses transformations which depend on dependent variables and their derivatives up to finite order (this relation is closely related to Cartan's absolute equivalence). To each system of ODE's we assign a differential algebra which is a commutative associative algebra with a derivation. This algebra is equiped with a natural filtration. Our first result says that two systems are dynamically equivalent if and only if their differential algebras are isomorphic. Our second result states that two systems are statically equivalent if and only if their filtered differential algebras are isomorphic.

1. Introduction

Attempts to understand the algebraic and/or geometric structure of nonlinear differential equations began in the last century in the work of Pfaff, Darboux, Frobenius and, especially, Sophus Lie. At the beginning of this century a lot of new ideas appeared in the papers of Elie Cartan.

A modern attempt to geometrize a general set of differential equations along the lines of E. Cartan is presented in the recent book [BCGGG] and also in [VKL]. The approach in the later book is more algebraic, compared to the approach of Cartan. An algebraic framework called differential algebra was created by Ritt [R] for studying nonlinear ODE's. A basic tool used in this approach is the notion of differential field. The ideas of Ritt were further developed along abstract algebraic lines by his collaborators (see Kolchin [Ko]).

For linear singular differential equations it turned particularly useful to replace an analytic object (a set of ODE's) by an algebraic one, i.e., a D-module.

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In this note we would like to present some results which indicate, in our opinion, that differential algebras should play for nonlinear ODE's a role analogous to the role of D-modules for linear differential equations. Namely, our results say that a differential algebra of a system contains all the information about the system, up to a natural equivalence relation.

To state our results we begin by introducing two equivalence relations in the set of nonlinear systems of ODE's.

2. Equivalence

A system of nonlinear ordinary differential equations can be written in the form

$$\Gamma\colon\ F_s\big(y, y^{(1)}, \ldots, y^{(p)}\big) = 0\,, \quad s = 1, \ldots, N\,,$$

where we assume that $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ are functions of time $t \in \mathbb{R}$, with the derivatives $y^{(i)} = d^i y/dt^i$. We consider systems of class C^k, i.e., we assume that $F \in C^k$, $k = \mathrm{pol}$, ω, or ∞, where pol stands for "polynomial".

By *behavior* $\mathcal{B}$ of Γ we mean the set of trajectories of Γ, that is all pairs (I, γ) of open (not necessarily bounded) intervals $I \subset \mathbb{R}$ and C^∞ functions $\gamma\colon I \to \mathbb{R}^n$ which satisfy equations Γ on I.

Equations Γ can be written in a more compact form if we introduce the following notation. Denote

$$Y = (y^0, y^1, y^2, \ldots) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \cdots = J$$

and identify $y^0 = y$. We think of Y as an element of the space J of infinite jets of functions $\mathbb{R} \to \mathbb{R}^n$. If $y = y(\cdot)$ is a function of time, then we identify $y^i = y^{(i)} = dy^i/dt^i$ and then $Y = Y(\cdot) = Jy$ is the infinite jet extension of $y(\cdot)$, where $Jy = (y, y^{(1)}, y^{(2)}, \ldots)$. With this notation we can write equations Γ in the form $F(Jy) = 0$ or, using our identification,

$$F(Y) = 0\,.$$

Here our function F depends on a finite number of variables parametrizing the space of infinite jets.

Let us consider another system $\tilde{\Gamma}$ with possibly different $\tilde{F}$, $\tilde{p}$, $\tilde{n}$ and $\tilde{N}$. Denote its behavior by $\tilde{\mathcal{B}}$.

Definition 1. Two systems Γ and $\tilde{\Gamma}$ of class C^k are called *statically equivalent* if there exist transformations of class C^k (called *static transformations*)

$$(ST)\colon\quad y = \chi(\tilde{y}), \quad \tilde{y} = \tilde{\chi}(y)\,,$$

such that the induced maps preserve the behaviors,

$$\chi_{\text{ind}}(\tilde{\mathcal{B}}) = \mathcal{B}, \quad \tilde{\chi}_{\text{ind}}(\mathcal{B}) = \tilde{\mathcal{B}},$$

(where χ_{ind} is the map induced from χ by composition of functions) and χ_{ind}, $\tilde{\chi}_{\text{ind}}$ are mutually inverse on $\mathcal{B}$ and $\tilde{\mathcal{B}}$.

Example 1. It is easy to see that the following system

$$(y_1^{(1)})^2 + 2y_1^{(1)}y_2 + (y_2)^2 = 0, \quad y_3 = 1,$$

is statically equivalent to the system

$$y_1^{(1)} + y_2 = 0, \quad y_3 = 1,$$

(via identity static transformation) and the latter one is statically equivalent to the 2–dimensional system

$$y_1^{(1)} + y_2 = 0.$$

As static feedback equivalence of control systems (cf., e.g., [Ku], [J2]) is a special case of static equivalence of systems in the form Γ, it follows from results of [Ku] and earlier results of this author that this equivalence is quite restrictive in case of underdetermined systems and, usually, leads to infinite dimensional (functional) invariants. For this reason, and for reasons coming from applications it is natural to consider a weaker equivalence relation.

Definition 2. Two systems Γ and $\tilde{\Gamma}$ of class C^k are called *dynamically equivalent* if there exist transformations of class C^k (called *dynamic transformations*)

$$(DT): \; y = \chi(\tilde{Y}), \quad \tilde{y} = \tilde{\chi}(Y),$$

which depend on finite jets, such that the induced maps preserve the behaviors,

$$\chi_{\text{ind}}(\tilde{\mathcal{B}}) = \mathcal{B}, \quad \tilde{\chi}_{\text{ind}}(\mathcal{B}) = \tilde{\mathcal{B}},$$

(here χ_{ind} is the map induced induced from χ by composition of functions, $\chi_{\text{ind}}(\gamma) = \chi \circ (J\gamma)$) and χ_{ind}, $\tilde{\chi}_{\text{ind}}$ are mutually inverse on $\mathcal{B}$ and $\tilde{\mathcal{B}}$.

Example 2. The last system in Example 1 is dynamically equivalent to the trivial ("free") 1–dimensional system which consists of a variable y with no equations whatsoever. The dynamic transformations which establish equivalence are of the form:

$$y_1 = y, \quad y_2 = -y^{(1)}, \quad \text{and} \quad y = y_1.$$

3. Differential algebras and criteria of equivalence

In order to state criteria of static and dynamic equivalence we introduce a differential algebra which corresponds to any system. In this paper by a differential algebra we mean a commutative associative algebra A with a linear operator $D : A \to A$ (called derivation) which commutes with multiplication by elements of $\mathbb{R}$ and satisfies the Leibnitz rule $D(ab) = (Da)b + a(Db)$, where $a, b \in A$. A homomorphism of differential algebras (A, D_A) and (B, D_B) is a homomorphism of algebras $h : A \to B$ which commutes with the derivations: $D_B h = h D_A$.

Let us consider the algebra $A^k = C^k(J, \mathbb{R})$ of functions of class C^k on the space of infinite jets J, each function depending on a finite order jet, only. There is a natural derivation of this algebra

$$D = \sum_{\substack{j=1,\dots,n,\\ i\geq 0}} y_j^{(i+1)} \frac{\partial}{\partial y_j^{(i)}}$$

and the pair (A^k, D) forms a differential algebra.

The system Γ defines the following differential ideal (ideal invariant under D) of (A^k, D)

$$I_\Gamma = I\{D^i F_s \,:\, i \geq 0,\ s = 1, \dots, N\},$$

where $I\{\dots\}$ stands for the ideal generated by elements within the brackets. The most naive (but sufficient for this paper) definition of the differential algebra of system Γ is the following.

Definition 3. The *differential algebra of system* Γ is the quotient differential algebra

$$A^k_\Gamma = A^k / I_\Gamma$$

with the induced derivation D_Γ defined by $D_\Gamma(a + I_\Gamma) = Da + I_\Gamma$.

With this definition of differential algebra of the system and no further assumption about the system it may happen that two systems displey the same behavior (the same solutions), but their differential algebras are not isomorphic. For example, this happens for the following two 1–dimensional systems.

Example 3. Consider two systems

$$\Gamma_1;\quad y^{(1)} - 1 = 0,$$
$$\Gamma_1;\quad (y^{(1)} - 1)^2 = 0.$$

The differential algebra of the first system is isomorphic to the algebra $C^k(\mathbb{R}^n, \mathbb{R})$, with the derivation $D = \partial/\partial y$ (and so has no zero divisors). The function $y^{(1)} - 1$ is a zero divisor in the algebra of the second system.

The above phenomenon can be eliminated either by a more adequate definition of the differential algebra of the system or by an additional assumption on the system. Here we introduce the following *rank assumption* which eliminates this phenomenon and guarantees sufficient regularity of solutions of system Γ:

$$(RA): \quad \operatorname{rank} \frac{\partial F}{\partial y^{(p)}} = N .$$

Note that this assumption implies that $n \geq N$. The second system in Example 3 does not satisfy (RA).

THEOREM 1. *Two systems Γ, $\tilde{\Gamma}$ of class C^k satisfying the rank assumption (RA) are dynamically equivalent if and only if their differential algebras (A^k_Γ, D_Γ) and $(A^k_{\tilde{\Gamma}}, D_{\tilde{\Gamma}})$ are isomorphic (as differential algebras).*

In order to state an analogous result for static equivalence we have to define a more "rigid" structure (comparing to the differential algebra of a system) as static equivalence is more restrictive than dynamic equivalence.

We define a sequence of subalgebras $\mathcal{F}_i \subset A^k$ as follows

$$\mathcal{F}_i = \left\{ a \in A^k \mid a \text{ depends on } i\text{-th order jet only} \right\}$$

(that is $a = a(y^{(0)}, \ldots, y^{(i)})$). We have that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots , \quad \bigcup_{i \geq 0} \mathcal{F}_i = A^k, \quad \text{and} \quad D\colon \mathcal{F}_i \to \mathcal{F}_{i+1} ,$$

and so $\mathcal{F} = \{\mathcal{F}_i\}_{i \geq 0}$ forms a filtration of (A^k, D). We define the quotient filtration

$$\mathcal{F}_i(\Gamma) = \mathcal{F}_i \cap I_\Gamma \subset A^k_\Gamma, \quad \mathcal{F}_\Gamma = \left\{\mathcal{F}_i(\Gamma)\right\}_{i \geq 0} ,$$

and verify easily that $D_\Gamma\colon \mathcal{F}_i(\Gamma) \to \mathcal{F}_{i+1}(\Gamma)$.

DEFINITION 4. The *filtered differential algebra of system* Γ is the triple

$$(A^k_\Gamma, \mathcal{F}_\Gamma, D_\Gamma) .$$

By *homomorphism of filtered differential algebras* we mean a homomorphism of differential algebras which preserves the filtrations (maps i-th element of the filtration into i-th element of the other filtration).

THEOREM 2. *Two systems* Γ, $\tilde{\Gamma}$ *of class* C^k *satisfying the rank assumption (RA) are statically equivalent if and only if their filtered differential algebras* $(A^k_\Gamma, \mathcal{F}_\Gamma, D_\Gamma)$ *and* $(A^k_{\tilde{\Gamma}}, \mathcal{F}_{\tilde{\Gamma}}, D_{\tilde{\Gamma}})$ *are isomorphic (as filtered differential algebras).*

Remark 1. For explicit systems (often called control systems) with two groups of variables $y = (x, u)$ and equations

$$\dot{x} = f(x, u)$$

the rank assumption is automatically satisfied. Then there is a straightforward way of "simplyfying" the differential algebra of the system by eliminating the variables $x^{(1)}, x^{(2)}, \dots$. Theorems 1 and 2 can then be deduced from the results of [J1] and [J2] (the general case will be proved elsewhere). Under mild additional assumptions these theorems can be specialized to give criteria ([J1], [J2]) for static and dynamic equivalence of control systems.

Remark 2. There is a way of obtaining similar results for systems not satisfying the rank assumption. This requires changing the definition of the differential algebra of the system and taking formal (instead of smooth) trajectories of the system in the definition of its behavior. In this generality, the results can be extended to systems of partial differential equations. This problem will be discussed in a future paper.

REFERENCES

[BCGGG] BRYANT, R. L.—CHERN, S. S.—GARDNER, R. B.—GOLDSCHMIDT, H. L.—GRIFFITHS, P. A.: *Exterior Differential Systems*, Springer, New York, 1991.

[J1] JAKUBCZYK, B.: *Dynamic equivalence of nonlinear control systems*, Preprint (1992).

[J1] JAKUBCZYK, B.: *Filtered differential algebras are complete invariants of static feedback*, Preprint 1114, Institute of Mathematics and its Applications, University of Minnesota, March 1993.

[Ko] KOLCHIN, E. R.: *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[Ku] KUPKA, I.: *On feedback equivalence*, in: Differential Geometry, Global Analysis, and Topology (A. Nicas and W. F. Shadwick, eds.), Canad. Math. Soc. Conf. Proc., Vol. 12, AMS 1991.

[R] RITT, J. F.: *Differential Algebra*, Amer. Math. Soc. Colloq. Publ., Vol. 33, Amer. Math. Soc., New York, 1950.

[VKL] VINOGRADOV, A. M.—KRASILSHCHIK, I. S.—LYCHAGIN, V. V.: *Introduction to Geometry of Nonlinear Differential Equations*, Nauka, Moscow, 1986.

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