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OSCILLATION THEORY OF SELF–ADJOINT EQUATIONS AND SOME ITS APPLICATIONS

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ABSTRACT. Oscillation properties of self-adjoint, even order, differential equations are investigated using the variational method. The results are used to study spectral properties of singular differential operators.

1. Introduction

In this contribution we deal with oscillation properties of the self-adjoint, two term equation

$$(-1)^n\big(r(t)y^{(n)}\big)^{(n)}+p(t)\,y=0\,, \tag{1}$$

where $t\in I=(a,b)$, $-\infty\le a<b\le\infty$, r^{-1}, $p\in L_{\mathrm{loc}}(I)$, $r(t)>0$. The literature covering the oscillation theory of self-adjoint equations is voluminous (recall at least the monographs [3, 12, 13, 18, 19]), so rather here the author's view on some aspects of the problem is presented.

First recall necessary definitions. Two points t_1, $t_2\in I$ are said to be *conjugate* relative to (1) if there exists a nontrivial solution of this equation for which $y^{(i)}(t_1)=0=y_2^{(i)}(t_2)$, $i=0,\dots,n-1$. Equation (1) is said to be *conjugate* on an interval $I_0\subseteq I$ if there exists a pair of points of I_0 which are conjugate relative to (1), in the opposite case (1) is said to be *disconjugate* on I_0. Equation (1) is said to be *oscillatory at b* if for any $c\in I$ (1) is conjugate on (c,b), in the opposite case it is said to be *nonoscillatory at b.*

These definitions are motivated by the calculus of variations. The following variational lemma elucidates this motivation and it is also the basic tool in the proof of the below given oscillation criteria.

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LEMMA 1. ([12]) *Equation (1) is conjugate on an interval* $I_0 = (c,d) \subseteq I$ *if and only if there exists a nontrivial function* $y \in W^{2,n}(I_0)$ *with* $\operatorname{supp} y \subset I_0$ *such that*

$$I(y;c,d) = \int_c^d \left[r(t)\left(y^{(n)}(t)\right)^2 + p(t)y^2(t) \right] dt \le 0 .$$

Now recall relation between general self-adjoint equation

$$\sum_{k=0}^{n} (-1)^k \left(p_k(t)\, y^{(k)}\right)^{(k)} = 0 \tag{2}$$

and linear Hamiltonian systems (further LHS). Let y be a solution of (2) and set $\boldsymbol{u} = \left(y, \dots, y^{(n-1)}\right)$, $v_n = p_n y^{(n)}$, $v_{n-k} = -v'_{n-k+1} + p_{n-k} y^{(n-k)}$, $k = 1, \dots, n-1$. Then $(\boldsymbol{u}, \boldsymbol{v})$ is a solution of the LHS

$$\boldsymbol{u}' = \mathbf{A}\boldsymbol{u} + \mathbf{B}(t)\boldsymbol{v}, \qquad \boldsymbol{v}' = \mathbf{C}(t)\boldsymbol{u} - \mathbf{A}^T\boldsymbol{v}, \tag{3}$$

where

$$\begin{aligned}
\mathbf{B}(t) &= \operatorname{diag}\{0, \dots, 0, p_n^{-1}(t)\}, \\
\mathbf{C}(t) &= \operatorname{diag}\{p_0(t), \dots, p_{n-1}(t)\}, \\
\mathbf{A} = A_{i,j} &= \begin{cases} 1, & \text{for } j = i+1, \quad i = 1, \dots, n-1, \\ 0, & \text{elsewhere}. \end{cases}
\end{aligned} \tag{4}$$

In this case we say that the solution $(\boldsymbol{u}, \boldsymbol{v})$ is generated by y. Simultaneously with (3) consider its matrix analogy

$$\mathbf{U}' = \mathbf{A}\mathbf{U} + \mathbf{B}(t)\mathbf{V}, \qquad \mathbf{V}' = \mathbf{C}(t)\mathbf{U} - \mathbf{A}^T\mathbf{V}, \tag{5}$$

where $\mathbf{U}, \mathbf{V}$ are $n \times n$ matrices. A solution $(\mathbf{U}, \mathbf{V})$ of (5) is said to be *isotropic* if $\mathbf{U}^T(t)\mathbf{V}(t) - \mathbf{V}^T(t)\mathbf{U}(t) \equiv 0$. An isotropic solution $(\mathbf{U}_b, \mathbf{V}_b)$ of (5) is said to be *principal at* b if $\mathbf{U}_b$ is nonsingular near b and

$$\lim_{t \to b-} \left(\int^t \mathbf{U}_b^{-1}(s)\, \mathbf{B}(s)\, \mathbf{U}_b^{T-1}(s)\, ds \right)^{-1} = 0 .$$

Let $(\mathbf{U}, \mathbf{V})$ be a solution of (2.3) which is linearly independent of $(\mathbf{U}_b, \mathbf{V}_b)$ (i.e., $(\mathbf{U}_b, \mathbf{V}_b)$, $(\mathbf{U}, \mathbf{V})$ form the base of the solution space of (5)), then $(\mathbf{U}, \mathbf{V})$ is said to be *nonprincipal at* b. The system $y_1, \dots, y_n$ of solutions of (2) is said to form the *principal* (*nonprincipal*) *system at* b if the solution $(\mathbf{U}, \mathbf{V})$ of the corresponding LHS (5) whose columns are generated by $y_1, \dots, y_n$ is principal (nonprincipal) at b. The principal (nonprincipal) system of solutions at b exists whenever (2) is nonoscillatory at b.

2. Oscillation criteria

In view of Lemma 1, for oscillation of (1) at b, the function p has to be "sufficiently negative" near b. What does it mean precisely is given in the following three oscillation criteria. In these criteria equation (1) is essentially regarded as a perturbation of the one-term equation

$$\left(r(t)y^{(n)}\right)^{(n)} = 0\,, \tag{6}$$

and negativity of p is "measured"by means of solutions of (6).

THEOREM 1. ([9]) *Let $y_1,\dots,y_n$ be the principal system of solutions at b of (6), $\mathbf{c}=(c_1,\dots,c_n)^T\in\mathbb{R}^n$ and $h=c_1y_1+\dots+c_ny_n$. If*

$$\int^b p(t)\,h^2(t)\,dt = -\infty\,, \tag{7}$$

then equation (1) is oscillatory at b.

Condition (7) is far from being necessary for oscillation of (1) as shows the simple example of the second order equation $-y''-\mu t^{-2}y=0$, $\mu>\frac{1}{4}$. The following two theorems deal with the case when the integral in (7) is convergent (like in the above example) or is divergent but at least one of solutions in the linear combination which define h is not from principal system.

THEOREM 2. ([5]) *Let $y_1,\dots,y_n$, $\mathbf{c}$ and h be the same as in Theorem 1. If*

$$\limsup_{t\to b}\frac{\int_t^b p(s)\,h^2(s)\,ds}{\mathbf{c}^T\left(\int^t \mathbf{U}^{-1}(s)\mathbf{B}(s)\mathbf{U}^{T-1}(s)\,ds\right)^{-1}\mathbf{c}} < -1\,, \tag{8}$$

where $\mathbf{U}=(U_{ij})=\left(y_j^{(i-1)}\right)$ is the Wronski matrix of $y_1,\dots,y_n$ and $\mathbf{B}$ is given by (4), then (1) is oscillatory at b.

THEOREM 3. ([5]) *Let $\tilde y_1,\dots,\tilde y_n$ be a nonprincipal system of solutions at b of (6), $\mathbf{c}=(c_1,\dots,c_n)^T\in\mathbb{R}^n$, $h=c_1\tilde y_1+\dots+c_n\tilde y_n$. If*

$$\limsup_{t\to b}\frac{\int^t p(s)\,h^2(s)\,ds}{\mathbf{c}^T\left(\int_t^b \widetilde{\mathbf{U}}^{-1}(s)\mathbf{B}(s)\widetilde{\mathbf{U}}^{T-1}(s)\,ds\right)^{-1}\mathbf{c}} < -1\,, \tag{9}$$

where $\widetilde{\mathbf{U}} = (\widetilde{U}_{ij}) = (\tilde{y}_j^{(i-1)})$ *is the Wronski matrix of* $\tilde{y}_1, \dots, \tilde{y}_n$ *and* $\mathbf{B}$ *is given by* (4), *then* (1) *is oscillatory at* b.

Note that these oscillation criteria — in contrast to the majority of recent ones, see [11, 15] — do not require any sign restriction on the function p.

Proof of Theorems. Let $t_0 \in I$ be arbitrary. According to Lemma 1 it suffices to find a nontrivial function $y \in W^{2,n}(t_0, b)$, $\operatorname{supp} y \subset (t_0, b)$, for which $I(y; t_0, b) \le 0$. Chose $t_0 < t_1 < t_2 < t_3 < b$ sufficiently close to b and define

$$y(t) = \begin{cases} 0\,, & t \in [a, t_0]\,, \\ f(t)\,, & t \in [t_0, t_1]\,, \\ h(t)\,, & t \in [t_1, t_2]\,, \\ g(t)\,, & t \in [t_2, t_3]\,, \\ 0\,, & t \in [t_3, b)\,, \end{cases}$$

where f, g are the solutions of (6) satisfying the boundary conditions

$$f^{(i)}(t_0) = 0\,, \quad f^{(i)}(t_1) = h^{(i)}(t_1)\,, \quad g^{(i)}(t_2) = h^{(i)}(t_2)\,, \quad g^{(i)}(t_3) = 0\,, \qquad i = 1, \dots, n\,.$$

After some calculation we get

$$\begin{aligned} &I(y; t_0, t_3) = \\ &= \int_{t_0}^{t_1} r(t)\big(f^{(n)}(t)\big)^2\,dt + \int_{t_1}^{t_2} r(t)\big(h^{(n)}(t)\big)^2\,dt + \int_{t_2}^{t_3} r(t)\big(g^{(n)}(t)\big)^2\,dt + \\ &\quad + \int_{t_0}^{t_1} p(t) f^2(t)\,dt + \int_{t_1}^{t_2} p(t)\,h^2(t)\,dt + \int_{t_2}^{t_3} p(t)\,g^2(t)\,dt = \\ &= \mathbf{c}^T\Big(\int_{t_0}^{t_1} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1} ds\Big)^{-1}\mathbf{c} + \mathbf{c}^T\Big(\int_{t_2}^{t_3} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1} ds\Big)^{-1}\mathbf{c} + \\ &\quad + \int_{t_0}^{t_1} p(t) f^2(t)\,dt + \int_{t_1}^{t_2} p(t)\,h^2(t)\,dt + \int_{t_2}^{t_3} p(t)\,g^2(t)\,dt\,. \end{aligned} \tag{10}$$

First suppose that $p(t) \le 0$ near b. Since $y_1, \dots, y_n$ is principal system at b, we have $\lim\limits_{t\to b} \mathbf{c}^T(\int^t \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1} ds)^{-1}\mathbf{c} = 0$. Now in the setting of Theorem 1, in view of (7), t_2 can be chosen such that

$$I(y;t_0,b) \le \mathbf{c}^T\left(\int_{t_0}^{t_1} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c} + \int_{t_1}^{t_2} p(t)\,h^2(t)\,dt + \varepsilon < 0\,,$$

whereby $t_3 > t_2$ is such that $\mathbf{c}^T\left(\int_{t_2}^{t_3} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c} < \varepsilon$.

In case of Theorem 2, we write (10) in the form

$$I(y;t_0,b) \le \mathbf{c}^T\left(\int_{t_0}^{t_1} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c}\times$$

$$\times\left[1+\frac{\int_{t_1}^{t_2} p(t)\,h^2(t)\,dt}{\mathbf{c}^T\left(\int_{t_0}^{t_1} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c}}+\frac{\mathbf{c}^T\left(\int_{t_2}^{t_3} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c}}{\mathbf{c}^T\left(\int_{t_0}^{t_1} \mathbf{U}^{-1}\mathbf{B}\mathbf{U}^{T-1}ds\right)^{-1}\mathbf{c}}\right].$$

If (8) holds, t_1 approaches b and $t_3 > t_2 > t_1$ are sufficiently close to b, the expression in the square brackets in the last inequality is negative.

To prove Theorem 3, in computation of $\int_{t_0}^{t_1} r(t)\big(f^{(n)}(t)\big)^2\,dt$, $\int_{t_2}^{t_3} r(t)\big(g^{(n)}(t)\big)^2\,dt$ we replace the matrix $\mathbf{U}$ by $\widetilde{\mathbf{U}}$ and (10) by

$$I(y;t_0,b) \le \mathbf{c}^T\left(\int_{t_2}^{t_3} \widetilde{\mathbf{U}}^{-1}\mathbf{B}\widetilde{\mathbf{U}}^{T-1}ds\right)^{-1}\mathbf{c}\times$$

$$\times\left[\frac{\int_{t_1}^{t_2} p(t)\,h^2(t)\,dt}{\mathbf{c}^T\left(\int_{t_2}^{t_3} \widetilde{\mathbf{U}}^{-1}\mathbf{B}\widetilde{\mathbf{U}}^{T-1}ds\right)^{-1}\mathbf{c}}+\frac{\mathbf{c}^T\left(\int_{t_0}^{t_1} \widetilde{\mathbf{U}}^{-1}\mathbf{B}\widetilde{\mathbf{U}}^{T-1}ds\right)^{-1}\mathbf{c}}{\mathbf{c}^T\left(\int_{t_2}^{t_3} \widetilde{\mathbf{U}}^{-1}\mathbf{B}\widetilde{\mathbf{U}}^{T-1}ds\right)^{-1}\mathbf{c}}+1\right].$$

If t_2 goes to b, (9) implies $I(y;t_0,b) < 0$.

If p oscillates near b and hence the last two integrals in (10) cannot be neglected, we proceed as follows. If the functions f/h, g/h are monotonic on (t_0,t_1), (t_2,t_3), respectively, by the second mean value theorem of integral calculus there exist $\xi_1 \in (t_0,t_1)$, $\xi_2 \in (t_2,t_3)$ such that

$$\int_{t_0}^{t_1} pf^2dt = \int_{t_0}^{t_1} ph^2(f/h)^2dt = \int_{\xi_1}^{t_1} ph^2dt, \qquad \int_{t_2}^{t_3} pg^2dt = \int_{t_2}^{\xi_2} ph^2dt\,.$$

Consequently, in Theorem 1 $\int\limits_{t_1}^{t_2} ph^2dt + \int\limits_{t_2}^{t_3} pg^2dt = \int\limits_{t_1}^{\xi_2} ph^2dt$ and (7) again implies $I(y; t_0, b) < 0$. In the proofs of Theorems 2, 3 we proceed similarly.

Monotonicity of $(f/h), (g/h)$ is proved via the transformation $y = hu$ which transforms (2) into an equation with the property that $u' = (y/h)'$ has at most $2n-2$ zeros (counting multiplicity) on I. Since $(f/h)', (g/h)'$ have zero points of multiplicity $n-1$ at t_0, t_1 and t_2, t_3, respectively, we have $(f/h)' \neq 0$, $t \in (t_0, t_1)$, $(g/h)' \neq 0$, $t \in (t_2, t_3)$. This implies required monotonicity and completes the proof. □

3. Modifications and examples

i) Let us pass from the two-term equation (1) to the general self-adjoint equation

$$M(y) + p(t)\, y = 0\,, \tag{11}$$

where $M(y) = \sum\limits_{k=0}^{n} (-1)^k \big(p_k(t) y^{(k)}\big)^{(k)}$. We shall show how to extend Theorem 2 to this more general situation, Theorems 1 and 3 extend in a similar way.

THEOREM 4. ([5]) *Suppose that equation $M(y) = 0$ is nonoscillatory at b and $y_1, \dots, y_n$ is its principal system of solutions at b. If $p(t) \le 0$ near b and*

$$\liminf_{t\to b} \frac{\int\limits_t^b p(s)\, h^2(s)\, ds}{\boldsymbol{c}^T \Big(\int\limits^t \mathbf{U}^{-1}(s)\mathbf{B}(s)\mathbf{U}^{T-1}(s)\, ds\Big)^{-1} \boldsymbol{c}} < -1\,, \tag{12}$$

where $\boldsymbol{c}$, h, $\mathbf{U}$ are the same as in Theorem 2 and $\mathbf{B} = \operatorname{diag}\{0, \dots, 0, p_n^{-1}\}$, then equation (11) is oscillatory at b. Moreover, if there exists $d \in I$ such that every solution of $M(y) = 0$ has at most $2n-1$ zeros on (d, b) and lim inf *in (12) is replaced by* lim sup, *the statement remains valid without any sign restriction on the function p.*

ii) In all previous criteria the test function h was a solution of (6) or of $M(y) = 0$. It is natural to ask whether some other test functions may be used. The answer is affirmative as it is shown in [8]. If $p(t) \le 0$ near b, a relatively large class of functions h may be used. If no sign restriction on p is assumed, we need the same assumption in equation $M(y) = 0$ as in the second part of Theorem 4 (this requirement equation (6) automatically satisfies). Moreover, the test function must in a certain way compare with solutions of equation

$M(y) = 0$. This modification enables us to prove several oscillation criteria originally proved only for nonpositive functions p, particularly those given in [11, 15], without sign restriction on p.

iii) The method used in the proofs of oscillation criteria from the previous section may be used to study sufficient conditions for the existence of at least two conjugate points in a given interval. This problem was, among others, investigated in [2, 4, 6, 7, 16, 17, 20]. Typical result is given in the following theorem.

THEOREM 5. ([4]) *Suppose that $y_1, \dots, y_m$, $1 \le m \le n$, are solutions of (6) which are contained both at principal systems of solutions at a and b. If there exist $c_1, \dots, c_m \in \mathbb{R}$ such that*

$$\int_a^b p(t)\, h^2(t)\, dt < 0 ,$$

where $h = c_1 y_1 + \cdots + c_m y_m$, then there exists at least one pair of points of $I = (a, b)$ which are conjugate relative to (1).

EXAMPLES.

1. Consider the equation

$$(-1)^{(n)} y^{(2n)} + p(t) y = 0 \tag{13}$$

as a perturbation of the Euler equation $y^{(2n)} - \mu_n t^{-2n} y = 0$, where $\mu_n = P\left(\frac{2n-1}{2}\right)$ is the so-called Kneser constant and $P(\lambda) = \lambda(\lambda-1)\cdots(\lambda-2n+1)$. Applying Theorem 4 with $h(t) = t^{\frac{2n-1}{2}}$ and $b = \infty$, we have

COROLLARY 1. ([10]) *Suppose that*

$$\limsup_{t\to\infty} \lg t \int_t^\infty s^{2n-1} \Big(p(s) + (-1)^n \frac{\mu_n}{s^{2n}} \Big)\, ds < -K_n ,$$

where

$$K_n = \frac{(-1)^n}{2} \frac{d^2}{d\lambda^2} P_{2n}(\lambda) \Big|_{\lambda = \frac{2n-1}{2}} ,$$

then (13) is oscillatory at ∞.

2. Let $\alpha \notin \{0, 1, \dots, 2n-1\}$ and consider the equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} + p(t) = 0 . \tag{14}$$

Modification of Theorem 2 from the part ii) of this section gives

COROLLARY 2. ([8, 11]) *Let* $\alpha + \sigma < 2n - 1$. *If*

$$\limsup_{t\to\infty} t^{2n-1-\alpha-\sigma} \int_t^\infty p(s)s^\sigma\,dt < -B_{n,\alpha,\sigma} - \frac{\binom{\sigma/2}{n}^2 (n!)^2}{2n-1-\alpha-\sigma},$$

$B_{n,\alpha,\sigma}$ *being a nonnegative real constant depending on* n, α, σ, *then* (14) *is oscillatory at* ∞.

Note that under the additional assumption $p(t) \le 0$ for large t this statement was proved in [11], where also the precise value of $B_{n,\alpha,\sigma}$ may be found. The method introduced in [8] enables to drop this assumption.

3. As an example of application of Theorem 5, consider the equation (13) on $I = \mathbb{R} = (-\infty, \infty)$. Since $y_1 = 1, \dots, y_n = t^{n-1}$ form principal system of $y^{(2n)} = 0$ both at $-\infty$ and ∞ (i.e., $n = m$), we have

COROLLARY 3. *If there exist* $c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\int_{-\infty}^{\infty} p(t)(c_1 + c_2 t + \cdots + c_n t^{n-1})^2\,dt < 0$$

then there exists at least one pair of points in $\mathbb{R}$ *which are conjugate relative to* (13).

Observe that Theorem 5 does not apply to (13) considered on $I = (0, \infty)$ since principal systems of $y^{(2n)} = 0$ at 0 and ∞ have no common solution. However, using the idea of Example 1, we have

COROLLARY 4. *Suppose*

$$\int_0^\infty t^{2n-1}\left(p(t) + (-1)^n \frac{\mu_n}{t^{2n}}\right) dt < 0$$

then there exists at least one pair of points in I *which are conjugate relative to* (13).

4. Application

In this section we mention one application of oscillation theory of self-adjoint equations in spectral theory of singular differential operators. Let $w \in L_{\text{loc}}(a, b)$ be a positive weight function and consider the operator

$$\ell(y) = \frac{(-1)^n}{w(t)}\left(r(t)y^{(n)}\right)^{(n)}, \qquad t \in I = [a, b)$$

in the weighted Hilbert space $L^2_w(I) = \{y | \int_a^b wy^2\,dt < \infty\}$. We suppose that a is the regular point (i.e., $a > -\infty$ and r^{-1}, w are integrable near a) and b is the singular point.

We say that operator ℓ has *property BD* if every self-adjoint extension of the minimal differential operator generated by ℓ has spectrum discrete and bounded below. For investigation of this property the following statement plays crucial role.

LEMMA. ([12]) *Operator ℓ has property BD if and only if the equation $\ell(y) = \lambda y$ is nonoscillatory at b for every $\lambda \in \mathbb{R}$.*

For the sake of comparison, recall the classical result of T k a c h e n k o and L e w i s.

THEOREM 6. ([12, 14]) *Let $b = \infty$ and $w \equiv 1$. Operator ℓ has property BD if and only if*

$$\lim_{t\to\infty} t^{2n-1} \int_t^\infty r^{-1}(s)\,ds = 0\,. \tag{15}$$

Application of Theorem 2 gives the following necessary condition for property BD of the general one-term operator ℓ.

THEOREM 7. *Let $y_1, \dots, y_n$ be the principal system of solutions at b of the equation $\left(w^{-1}(t)y^{(n)}\right)^{(n)} = 0$, $\mathbf{U}$ be their Wronski matrix and $\overline{\mathbf{B}} = \operatorname{diag}\{0, \dots, 0, w\}$. If ℓ has property BD then*

$$\lim_{t\to b} \frac{\int_t^b r^{-1}(s)\left(c_1y_1(s) + \dots + c_ny_n(s)\right)^2 ds}{\mathbf{c}^T\left(\int_t \mathbf{U}^{-1}(s)\overline{\mathbf{B}}(s)\mathbf{U}^{T-1}(s)\,ds\right)^{-1}\mathbf{c}} = 0 \tag{16}$$

for every $\mathbf{c} = (c_1, \dots, c_n)^T \in \mathbb{R}^n$.

Setting $b = \infty$, $w \equiv 1$, $y_1 = 1, \dots,$ $y_n = t^{n-1}$, $\mathbf{c} = e_1 = (1, 0, \dots, 0)^T$, it is not difficult to verify that (15) is a special case of (16). In [1, 5] it was proved that for certain class of weight functions (16) is also sufficient for property BD, however, for general weight functions w this problem remains open.

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