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APPROXIMATE TRAJECTORIES AND LYAPUNOV EXPONENTS FOR DYNAMICAL SYSTEMS

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ABSTRACT. The following problems arising in the investigation of dynamical systems by pseudotrajectories are discussed: shadowing and weak shadowing, approximate evaluation of Lyapunov exponents, approximation of the shape of attractors.

Let M be a compact, C^∞-smooth n-dimensional manifold with Riemannian metric d. We consider the space $Z(M)$ of discrete dynamical systems generated by homeomorphisms $\phi\colon M \to M$ with the C^0-topology induced by the metric

$$\rho_0(\phi,\psi) = \max_{x\in M}\Bigl(d\bigl(\phi(x),\,\psi(x)\bigr),\, d\bigl(\phi^{-1}(x),\,\psi^{-1}(x)\bigr)\Bigr).$$

It is well-known that $Z(M)$ is a complete metric space. We denote below by $O(x,\phi)$ the trajectory of a point $x\in M$ with respect to $\phi\in Z(M)$:

$$O(x,\phi) = \{\phi^k(x)\colon k\in\mathbb{Z}\}.$$

Fix $\delta>0$. We say that a set of points $\xi=\{x_k\colon k\in\mathbb{Z}\}$ or $\xi=\{x_k\colon k\geq 0\}$ is a δ-trajectory (pseudotrajectory) of ϕ if

$$d\bigl(x_{k+1},\phi(x_k)\bigr)<\delta,\quad k\in\mathbb{Z}\ (k\geq 0).$$

Pseudotrajectories are a common idealization of "locally accurate" numerical methods for dynamical systems. We can consider a numerical method of accuracy $\delta>0$ for a system $\phi\in Z(M)$ as a map $\psi\colon M\to M$ such that

$$d\bigl(\phi(x),\,\psi(x)\bigr)<\delta,\quad x\in M. \tag{1}$$

Evidently, if (1) holds, then for any $x\in M$ the set $\xi=\{\psi^k(x)\colon k\geq 0\}$ is a δ-trajectory of ϕ.

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1. Shadowing

We say that $\phi \in Z(M)$ has the POTP (*pseudoorbit tracing property*) if given $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-trajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ there exists $x \in M$ with

$$d(\phi^k(x), x_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

We say that $\phi \in Z(M)$ has the WSP (*weak shadowing property*) if given $\varepsilon > 0$ there is $\delta > 0$ such that for any δ-trajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ there exists $x \in M$ with

$$\xi \subset N_\varepsilon(O(x, \phi)) \tag{2}$$

(here $N_\varepsilon(A)$ is the ε-neighborhood of a set $A \subset M$).

As usual we say that a subset of a topological space is residual if it contains a countable intersection of open and dense subsets. We say that a generic system $\phi \in Z(M)$ satisfies property $\mathcal{R}$ if there is a residual subset of $Z(M)$ such that any system in this subset satisfies $\mathcal{R}$.

K. Odani in [1] showed that if $\dim M \le 3$ then a generic $\phi \in Z(M)$ has the POTP.

Theorem 1 [2]. *A generic system $\phi \in Z(M)$ (with arbitrary $\dim M$) has the WSP.*

Let us give an example of a system ϕ which has the WSP and does not have the POTP. Consider $M = S^1$ with coordinate $x \in \mathbb{R}$ (mod 1). Let $\phi(x) = x + \alpha$ (mod 1) where α is irrational, then for any $x \in M$ its trajectory is dense in M. Hence, for any δ-trajectory ξ, for any $x \in M$, and for any $\varepsilon > 0$ (1) holds, so that ϕ has the WSP. Take arbitrary $\delta > 0$ and $\beta \in \mathbb{R}$ with $0 < |\alpha - \beta| < \delta$. Consider $\psi(x) = x + \beta$ (mod 1). Evidently, (1) holds, and for any x, y there is $k \in \mathbb{Z}$ such that

$$d(\phi^k(x), \psi^k(y)) \ge \frac{1}{2},$$

hence ϕ does not have the POTP.

Now let ϕ be a diffeomorphism of class C^1, and let $D\phi(x)$ be the derivative of ϕ at x. Denote by T_xM the tangent space of M at x, and by $|v|$ the norm of $v \in T_xM$ generated by d. For $x \in M$ define

$$L^+(x) = \{v \in T_xM : |D\phi^k(x)v| \to 0 \quad \text{as} \quad k \to \infty\},$$
$$L^-(x) = \{v \in T_xM : |D\phi^k(x)v| \to 0 \quad \text{as} \quad k \to -\infty\}.$$

We say that ϕ satisfies the STC (*strong transversality condition*) if for any $x \in M$ we have $L^+(x) + L^-(x) = T_xM$. It is well known (see [3]) that ϕ satisfies the STC if and only if ϕ is structurally stable.

Theorem 2 [4]. *If a diffeomorphism ϕ satisfies the STC then there exist $L, \Delta > 0$ such that if $\xi = \{x_k : k \in \mathbb{Z}\}$ is a δ-trajectory with $\delta < \Delta$ then there is a point $x \in M$ with*

$$d(\phi^k(x), x_k) \le L\delta.$$

This theorem is a generalization of results of C. Robinson [5], K. Sawada [6].

2. Lyapunov exponents

Let ϕ be a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Consider two maps $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$, $\Psi\colon \mathbb{R}^n \to \mathscr{L}_n$ ($\mathscr{L}_n$ is the space of $n \times n$ matrices). For a point $x \in \mathbb{R}^n$ we consider the upper Lyapunov exponent of $O(x, \phi)$:

$$\mu(x) = \max_{\substack{v\in\mathbb{R}^n\\|v|=1}} \overline{\lim_{m\to\infty}} \frac{1}{m} \log\left|D\phi^m(x)v\right|,$$

and the approximate exponent

$$\tilde{\mu}(x) = \max_{\substack{v\in\mathbb{R}^n\\|v|=1}} \overline{\lim_{m\to\infty}} \frac{1}{m} \log\left|\Phi(x_{m-1})\cdots\Phi(x_0)v\right|,$$

here $x_k = \psi^k(x)$. Assume that for some $\delta > 0$ we have

$$\left|\phi(x) - \psi(x)\right| < \delta, \qquad \left\|\Phi(x) - D\phi(x)\right\| < \delta, \tag{3}$$

in this case the pair (ψ, Ψ) is a model of a numerical method with accuracy δ of evaluating of Lyapunov exponents.

THEOREM 3 [7]. *Assume that $\phi \in C^2$, and that Λ is a hyperbolic attractor of ϕ with one-dimensional unstable foliation. Then there exist $L, \Delta > 0$ such that if (3) holds with $\delta < \Delta$ then for any point $x \in N_\delta(\Lambda)$ there is a point $y \in \Lambda$ with*

$$\left|\tilde{\mu}(x) - \mu(y)\right| \le L\delta .$$

3. Shape of attractors

Let I be an attractor of $\phi \in Z(M)$, denote by J the boundary of I and by $D(I)$ its basin of attraction. There exists a neighborhood V of I such that

$$I \subset V \subset \overline{V} \subset D(I), \qquad \phi(\overline{V}) \subset V .$$

In this case V is called an absorbing neighborhood of I. P. Kloeden and J. Lorenz [8] showed (in a slightly different situation — for one-step discretizations of systems of differential equations) that if V is an absorbing neighborhood of an attractor I and $\xi = \{x_k\colon k \in \mathbb{Z}\}$ is a δ-trajectory with small δ then V absorbs ξ, that is $x_k \in V$ implies $x_{k+1} \in V$.

We say that $\Xi = \{\xi(p)\colon p \in M\}$ is a $CF(\delta,\phi)$ (complete family of δ-trajectories for ϕ) if any $\xi(p) = \{x_k(p)\colon k \geq 0\}$ is a δ-trajectory with $x_0(p) = p$.

Denote by $R(A,B)$ the Hausdorff distance between two compact sets A, B. Let V be an absorbing neighborhood of an attractor I, consider the set

$$F = \overline{V} \setminus \phi(V).$$

Let for natural T and for a $CF(\delta,\phi)$ $\Xi = \{\xi(p)\}$ where $\xi(p) = \{x_k(p)\colon k \geq 0\}$,

$$\Xi(T,F) = \bigcup_{p\in F} x_T(p), \qquad \Xi(T,\infty,F) = \bigcup_{k\geq T} \Xi(k,F).$$

THEOREM 4. *Consider arbitrary $\phi \in Z(M)$. Given $\varepsilon > 0$ there exists $T(\varepsilon)$ such that for any $T \geq T(\varepsilon)$ we can find $\delta(T) > 0$ with the property: if Ξ is a $CF(\delta,\phi)$ with $\delta \in \big(0,\delta(T)\big)$ then*

$$R\big(\Xi(T,F),J\big) < \varepsilon.$$

THEOREM 5. *Assume that either $I = J$ or ϕ has the WSP. Then given $\varepsilon > 0$ there exists $T(\varepsilon)$ such that for any $T \geq T(\varepsilon)$ we can find $\delta(T) > 0$ with the property: if Ξ is a $CF(\delta,\phi)$ with $\delta \in \big(0,\delta(T)\big)$ then*

$$R\big(\Xi(T,\infty,F),J\big) < \varepsilon.$$

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