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ATTRACTORS OF NON–AUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS AND THEIR DIMENSION

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ABSTRACT. We have studied uniform attractors of non-autonomous nonlinear partial differential equation with almost periodic (a.p.) symbols. We have proved the attractor existence theorem for the 2D Navier–Stokes system with a.p. in time external forces, for the reaction-diffusion system and for the dissipative hyperbolic equation with a.p. in time terms. When symbols are quasiperiodic (q.p.) in time functions, we present the upper bounds for the Hausdorff dimension of uniform attractors of above problems arising in mathematical physics.

Introduction

Dynamical systems corresponding to autonomous evolution equations and their attractors have been studied intensively in mathematical literature especially in the last decade (see, for examples, books [14], [2] and the literature cited there). The non-autonomous infinite-dimensional dynamical systems are less understood. Such systems have been considered in the works [13], [6], [8], and others. Some general constructions and notions (for example, processes and skew product flows) have been presented in these works with applications mostly to ordinary differential equations and some functional equations. The book [10] contains the systematic study of attractors anduniform attractors of processes, i.e., two-parametric families of operators describing non-autonomous system with applications to some classes of partial differential equations.

In the paper we present a rather simple approach to the investigation of non-autonomous infinite-dimensional systems. We think, that this method is well-suited to the analysis of problems arising in mathematical physics. Using this method, we have proven attractor existence theorems for the classes of

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non-autonomous evolution equations that have terms as general as the terms of the corresponding classes of autonomous evolution equations. Also, the method permits to estimate the Hausdorff dimension of attractors of non-autonomous evolution equations. The approach is based on the analysis of a functional parameter of an equation called time symbol.

1. Uniform attractors of families of processes

Let E be a Banach space, and a two-parametric family of mappings $\{U(t,\tau)\} = \{U(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R}\}$ acts on E: $U(t,\tau)\colon E \mapsto E$, $t \geq \tau$, $\tau \in \mathbb{R}$.

Definition 1. A two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ is said to be a process in E if the following conditions hold:

i) $U_\sigma(t,s) \cdot U_\sigma(s,\tau) = U_\sigma(t,\tau)$, $\forall t \geq s \geq \tau$, $\tau \in \mathbb{R}$,

ii) $U_\sigma(\tau,\tau) = I$ is the identity operator, $\tau \in \mathbb{R}$.

By $\mathcal{B}(E)$ we denote the collection of sets bounded in E. Consider a family of processes $\{U_\sigma(t,\tau)\}$ depending on a functional parameter $\sigma \in \Sigma$. The parameter σ is said to be the symbol of the process $\{U_\sigma(t,\tau)\}$ and the set Σ is said to be the symbol space. In the sequel, Σ is assumed to be a complete metric space.

Definition 2. A set $B_0 \in E$ is said to be *uniformly* (w.r.t. $\sigma \in \Sigma$) *absorbing for the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(E)$ there exists $T = T(\tau, B) \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subseteq B_0$ $\forall t \geq T$.

Definition 3. A set P belonging to E is said to be *uniformly* (w.r.t. $\sigma \in \Sigma$) *attracting for the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if $\sup_{\sigma\in\Sigma} \operatorname{dist}_E\big(U_\sigma(t,\tau) B, P\big) \to 0$ $(t \to +\infty)$ for any $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(E)$. Recall that $\operatorname{dist}_E(X,Y) = \sup_{x\in X} \operatorname{dist}_E(x,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_E$.

A family of processes possessing a compact uniformly absorbing set is said to be uniformly compact, and one possessing a compact uniformly attracting set, uniformly asymptotically compact.

Definition 4. A closed set $\mathcal{A}_\Sigma \subset E$ is said to be the *uniform* (w.r.t. $\sigma \in \Sigma$) *attractor of the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if it is uniformly (w.r.t. $\sigma \in \Sigma$) attracting (attracting property), and it is contained in any closed uniformly (w.r.t. $\sigma \in \Sigma$) attracting set $\mathcal{A}'_\Sigma$ of the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$: $\mathcal{A}_\Sigma \subseteq \mathcal{A}'_\Sigma$ (property of minimality).

Theorem 1. *If a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, is uniformly asymptotically (w.r.t. $\sigma \in \Sigma$) compact then it possesses the uniform (w.r.t. $\sigma \in \Sigma$) compact attractor $\mathcal{A}_\Sigma$.*

2. On the reduction of families of processes to semigroups

Let a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, acts in a Banach space E. Suppose that the symbol space Σ is a complete metric space, and a certain invariant semigroup $\{T(t)\}_{t\geq 0}$ $(T(t_1)T(t_2) = T(t_1+t_2)$, $\forall t_1, t_2 \geq 0$, $T(0) = I)$ acts on it: $T(t)\Sigma = \Sigma \ \forall t \geq 0$. Let us assume that the following translation identity is valid:

$$U_\sigma(t+s, \tau+s) = U_{T(s)\sigma}(t, \tau) \quad \forall \sigma \in \Sigma, \ t \geq \tau, \ \tau \in \mathbb{R}, \ s \geq 0. \tag{2.1}$$

Let us construct the semigroup $\{S(t)\}$ acting on the extended phase space $E \times \Sigma$ and corresponding to the family of process $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, under the condition (2.1):

$$S(t)(u, \sigma) = \big(U_\sigma(t, 0)u, T(t)\sigma\big), \quad t \geq 0, \ (u, \sigma) \in E \times \Sigma. \tag{2.2}$$

Proposition 1. *The family of mappings $\{S(t)\}$ acting in $E \times \Sigma$ by the formula (2.2), under the conditions (2.1), forms a semigroup on $E \times \Sigma$: $S(t_1)S(t_2) = S(t_1 + t_2)$, $\forall t_1, t_2 \geq 0$, $S(0) = I$.*

Example 1. Non-autonomous evolution equation with a.p. symbol. Consider a family of non-autonomous evolution equations depending on a functional parameter $\sigma(t) \in C_b(\mathbb{R}, \mathcal{M})$, where $C_b(\mathbb{R}, \mathcal{M})$ denotes the space of bounded continuous functions on $\mathbb{R}$ with values in a certain complete metric space $\mathcal{M}$. These equations have the form:

$$\partial_t u = A_{\sigma(t)}(u), \quad t \in \mathbb{R}, \tag{2.3}$$

where, for any fixed $t \in \mathbb{R}$, $A_{\sigma(t)}(u)$ is a nonlinear operator acting from a Banach space E_1 into a Banach space E_0: $A_{\sigma(t)}(.)$: $E_1 \mapsto E_0$. Usually the space E_1 is dense in E_0. A functional parameter $\sigma(t)$ belongs to a certain closed set Σ, $\Sigma \subset C_b(\mathbb{R}, \mathcal{M})$. The element $\sigma(t)$ is called the time symbol of the equation (2.3) or simply the symbol. The set Σ is called the symbol space.

Let $T(h)$ be a translation operator along the time-axis:

$$T(h)\sigma(t) = \sigma(t+h), \quad h \in \mathbb{R}.$$

We assume, that for any $\sigma(t) \in \Sigma$ the function $T(h)\sigma(t) = \sigma(t+h)$ belongs to Σ for any $h \in \mathbb{R}$. This implies that the symbol space Σ is strictly invariant under the action of the translation group $\{T(h),\, h \in \mathbb{R}\} : T(h)\Sigma = \Sigma \;\; \forall h \geq 0$. We supplement the equation (2.3) with the initial conditions at $t = \tau$, $\tau \in \mathbb{R}$:

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in E, \tag{2.4}$$

where E is a Banach space, $E_1 \subseteq E \subseteq E_0$. Let us assume that for any symbol $\sigma(t) \in \Sigma$ the problem (2.3), (2.4) is uniquely solvable for any $\tau \in \mathbb{R}$ and arbitrary $u_\tau \in E$. We shall specify the meaning of the expression "the function $u(t)$ is a solution of (2.3)" in each particular case. Let also $u(t) \in E$ for any $t \geq \tau$. Thus, $u(t)$ can be represented in the form: $u(t) = U_\sigma(t, \tau)u_\tau$, $\sigma = \sigma(t) \in \Sigma$, $u_\tau \in E$, $\tau \in \mathbb{R}$, $t \geq \tau$. It is easy to show, that the two-parametric family of mappings $\{U_\sigma(t, \tau),\, t \geq \tau,\, \tau \in \mathbb{R}\}$ forms a process corresponding to the problem (2.3), (2.4) with the time symbol $\sigma(t) = \sigma \in \Sigma$, $U_\sigma(t, \tau)\colon E \mapsto E$, $t \geq \tau$, $\tau \in \mathbb{R}$. The problem (2.3), (2.4) with an arbitrary symbol $\sigma \in \Sigma$ generates the family of processes $\{U_\sigma(t, \tau),\, t \geq \tau,\, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$. This family satisfies the translation identity (2.1). The translation identity follows from the uniqueness of the solution $u(t)$ of the problem (2.3), (2.4).

In this paper we study equation (2.3) with almost periodic (a.p.) in time t symbols $\sigma(t)$. Values of $\sigma(t)$ belong to a complete metric space $\mathcal{M}$. It is well-known, according to Bochner–Amerio criterion, that a.p. function $\sigma(t)$ possesses the following characteristic property: the set of all its translations $\{\sigma(t+h) = T(h)\sigma(t),\, h \in \mathbb{R}\}$ forms a precompact set in $C_b(\mathbb{R}, \mathcal{M})$ (see, for example, [1]). The closure in $C_b(\mathbb{R}, \mathcal{M})$ of this set is said to be the hull $\mathcal{H}(\sigma)$ of the function $\sigma(t)$:

$$\mathcal{H}(\sigma) = \overline{\{\sigma(t+h) = T(h)\sigma(t),\, h \in \mathbb{R}\}}^{C_b(\mathbb{R}, \mathcal{M})} \subset\subset C_b(\mathbb{R}, \mathcal{M}).$$

In the sequel, a symbol space Σ coincides with a hull $\mathcal{H}(\sigma_0)$ of some fixed a.p. function $\sigma_0 = \sigma_0(t)$, $\Sigma = \mathcal{H}(\sigma_0)$. If a function $\sigma_0(t)$ is a.p., then any function $\sigma(t) \in \mathcal{H}(\sigma_0)$ is a.p. too. Evidently, the argument translation group $\{T(h),\, h \in \mathbb{R}\}$ is strictly invariant on $\Sigma = \mathcal{H}(\sigma_0)$. By Proposition 1, the semigroup $\{S(t)\}$ corresponds to the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$. The semigroup $\{S(t)\}$ acts on $E \times \mathcal{H}(\sigma_0)$ by the formula:

$$S(t_1)(u_0, \sigma) = \big(U_\sigma(t_1, 0)u_0,\, \sigma(t+t_1)\big),\; u_0 \in E,\, \sigma \in \mathcal{H}(\sigma_0), \tag{2.5}$$

where $U_\sigma(t, 0)u_0 = u(t)$ is a solution of (2.3) with the symbol $\sigma(t)$ and the initial condition $u|_{t=0} = u_0 \in E$.

Example 2.

Non-autonomous evolution equation with quasiperiodic (q.p.) symbol. Consider Example 1 with $\sigma_0(t) = \varphi(\alpha_1 t, \ldots, \alpha_k t) = \varphi(\alpha t)$, where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $\varphi(\omega_1, \ldots, \omega_i + 2\pi, \ldots, \omega_k) = \varphi(\omega_1, \ldots, \omega_i, \ldots, \omega_k)$ is a 2π-periodic function in each argument ω_i, $\omega = (\omega_1, \ldots, \omega_k) \in \mathcal{T}^k$, $\varphi \in C(\mathcal{T}^k, \mathcal{M})$. Here $\mathcal{T}^k$ is k-dimensional torus. We assume that α_i $(i = 1, \ldots, k)$ are rationally independent numbers (otherwise, one can reduce the number of variables ω_i). Consider the symbol space $\Sigma = \mathcal{H}(\sigma_0)$. It can be shown, that $\mathcal{H}(\sigma_0) = \{\varphi(\alpha t + \omega_0),\ \omega_0 \in \mathcal{T}^k\}$. We can consider symbol space $\mathcal{T}^k$ instead of $\mathcal{H}(\sigma_0)$ taking in mind the continuous mapping from $\mathcal{T}^k$ into $\mathcal{H}(\sigma_0)$: $\omega_0 \mapsto \varphi(\alpha t + \omega_0)$. The translation group of the torus $\mathcal{T}^k$, $T(t)\omega_0 = [\omega_0 + \alpha t] \pmod{2\pi)^k}$, corresponds to the argument translation group on $\mathcal{H}(\sigma_0)$. Obviously, $T(t)\mathcal{T}^k = \mathcal{T}^k$ $\forall t \geq 0$. The semigroup $\{S(t)\}$ acts on $E \times \mathcal{T}^k$ by the formula:

$$S(t)(u_0, \omega_0) = \left(U_{\omega_0}(t, 0)u_0,\ [\omega_0 + \alpha t] \ (\mathrm{mod}\ 2\pi)^k\right), \quad u_0 \in E,\ \omega_0 \in \mathcal{T}^k, \quad (2.6)$$

where $u(t) = U_{\omega_0}(t, 0)u_0$ is a solution of (2.3) with q.p. symbol $\sigma(t) = \varphi(\alpha t + \omega_0)$ and the initial conditions $u|_{t=0} = u_0 \in E$.

Let us return to general families of processes.

Definition 5. A family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, acting in E is said to be $(E \times \Sigma, E)$*-continuous*, if for any fixed t and τ, $t \geq \tau$, $\tau \in \mathbb{R}$ the mapping $(u, \sigma) \mapsto U_\sigma(t, \tau)u$ is continuous from $E \times \Sigma$ into E.

Definition 6. A curve $u(s)$, $s \in \mathbb{R}$, is said to be a *complete trajectory of the process* $\{U(t, \tau)\}$, if $U(t, \tau)u(\tau) = u(t)$ $\forall t \geq \tau$, $\tau \in \mathbb{R}$.

Definition 7. The *kernel* $\mathcal{K}$ *of the process* $\{U(t, \tau)\}$ consists of all bounded complete trajectories of the process $\{U(t, \tau)\}$: $\mathcal{K} = \{u(.) \mid u(t), t \in \mathbb{R}, u(.)$ is a complete trajectory of $\{U(t, \tau)\}$, $\|u(t)\|_E \leq M_u$ $\forall t \in \mathbb{R}\}$. The set $\mathcal{K}(s) = \{u(s) \mid u(.) \in \mathcal{K}\} \subseteq E$ is said to be the *kernel section at time* $t = s$, $s \in \mathbb{R}$.

Consider two projectors Π_1 and Π_2 from $E \times \Sigma$ onto E and Σ respectively: $\Pi_1(u, \sigma) = u$, $\Pi_2(u, \sigma) = \sigma$.

Theorem 2. *Let a family of processes* $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, *acting in the space* E *be uniformly (w.r.t.* $\sigma \in \Sigma$*) asymptotically compact and* $(E \times \Sigma, E)$*–continuous. Also let* Σ *be a compact metric space,* $\{T(t)\}$ *be a continuous invariant* $(T(t)\Sigma = \Sigma\ \forall t \geq 0)$ *semigroup on* Σ *satisfying the translation identity (2.1). Then the semigroup* $\{S(t)\}$ *corresponding to the family of processes*

$\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *and acting on* $E \times \Sigma$ *(see (2.2)) possesses the compact attractor* $\mathcal{A}$, *which is strictly invariant with respect to* $\{S(t)\}$: $S(t)\mathcal{A} = \mathcal{A}$ $\forall\, t \geq 0$. *Moreover,*

i) $\Pi_1 \mathcal{A} = \mathcal{A}_1 = \mathcal{A}_\Sigma$ *is the uniform* (w.r.t. $\sigma \in \Sigma$) *attractor of the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$;

ii) $\Pi_2 \mathcal{A} = \mathcal{A}_2 = \Sigma$;

iii) $\mathcal{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\}$;

iv) $\mathcal{A}_1 = \mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0)$. *Here* $\mathcal{K}_\sigma(0)$ *is the section at time* $t = 0$ *of the kernel* $\mathcal{K}_\sigma$ *of the process* $\{U_\sigma(t,\tau)\}$ *with the symbol* $\sigma \in \Sigma$.

COROLLARY 1. *Under the conditions of Theorem 2, for any* $\sigma \in \Sigma$ *the kernel of the process* $\{U_\sigma(t,\tau)\}$ *is not empty. It means, that there exists at least one bounded complete trajectory of the process* $\{U_\sigma(t,\tau)\}$ *for any* $\sigma \in \Sigma$.

COROLLARY 2. *Let* $\sigma_0(t)$ *be a.p. (or q.p.) function and* $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, *be a family of processes generated by the problem (2.3), (2.4), where* $\sigma \in \mathcal{H}(\sigma_0) = \Sigma$. *Assume that* $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, *is uniformly (w.r.t.* $\sigma \in \Sigma$*) asymptotically compact and* $(E \times \mathcal{H}(\sigma_0), E)$*–continuous family of processes. Then the set*

$$\mathcal{A}_1 = \mathcal{A}_\Sigma = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0)$$

is the uniform attractor of the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, *where* $\mathcal{K}_\sigma(0)$ *is the kernel section at time* $t = 0$ *of the process* $\{U_\sigma(t,\tau)\}$.

3. Examples of non-autonomous evolution equations and systems having uniform attractors

a) Navier–Stokes system with an a.p. external force.

Consider 2D Navier–Stokes system after excluding the pressure:

$$\partial_t u + \nu L u + B(u,u) = \varphi(t), \quad x = (x_1, x_2) \in \Omega \subset\subset \mathbb{R}^2,$$

$$\varphi(t) = \varphi(x,t),\ L = -\Pi\Delta,\ B(u,u) = \Pi \sum_{i=1}^{2} u_i \partial_{x_i} u,\ u|_{\partial\Omega} = 0, \tag{3.1}$$

$u = u(x,t) = \left(u^{(1)}(x,t), u^{(2)}(x,t)\right)$, $\varphi = \left(\varphi^{(1)}, \varphi^{(2)}\right)$. (See [15], [11]). By H (H_1) we denote the closure in the $\left(L_2(\Omega)\right)^2$ $\left(\left(H_1(\Omega)\right)^2\right)$ norm $\|\cdot\|$ $(\|\cdot\|_1)$ of the set $\mathcal{V}_0 = \{v \colon v \in (C_0^\infty(\Omega))^2,\ (\nabla, v) = 0\}$, by Π we denote the orthogonal projector on H (in $(L_2(\Omega))^2$) and its different extensions. The function $\varphi(t) = \varphi(\cdot\,, t)$ is assumed to belong to $C_b(\mathbb{R},\ H)$. The initial conditions are posed at $t = \tau,\ \tau \in \mathbb{R}$:

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in H\,. \tag{3.2}$$

The problem (3.1), (3.2) has a unique solution $u(t) \in C_b([\tau, +\infty), H) \cap L_2((\tau, T), H_1)$ $\forall T \geq \tau,\ t \geq \tau$, and $\partial_t u \in L_2((\tau, T), H_{-1})$, where $H_{-1} = (H_1)^*$. Now assume that $\varphi(t)$ in (3.1) is a.p. function with values in H. Let $\mathcal{H}(\varphi)$ be the hull of φ in $C_b(\mathbb{R}, H)$. Consider the family of Cauchy problems (3.1), (3.2) where $\varphi(x, t)$ is replaced by any function $g(x, t) \in \mathcal{H}(\varphi)$. Obviously, for all $g \in \mathcal{H}(\varphi)$ the problem (3.1), (3.2) has a unique solution $u(t)$. Thus, the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(\varphi)$, acting in H, corresponds to the problem. The time symbol $\sigma(t)$ of the equation (3.1) is the function $g(x, t)$, $g(\cdot\,, t) = \sigma(t)$. The symbol space Σ is $\mathcal{H}(\varphi)$. By the assumption, $\Sigma = \mathcal{H}(\varphi) \subset\subset C_b(\mathbb{R}, H)$. The family of processes is uniformly (w.r.t. $g \in \mathcal{H}(\varphi)$) compact, and $(H \times \mathcal{H}(\varphi), H)$–continuous (see [3], [4]). Now let $\{S(t)\}$ be the semigroup acting on $H \times \mathcal{H}(\varphi)$ according to the formula (2.5), the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(\varphi)$, satisfies all the conditions of Theorem 3.2. It follows that the semigroup $\{S(t)\}$ possesses the compact in $H \times \mathcal{H}(\varphi)$ attractor $\mathcal{A}$. Moreover, the set $\Pi_1\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_{\mathcal{H}(\varphi)}$ $(\Pi_1(u, g) = u)$ is the uniform (w.r.t. $g \in \mathcal{H}(\varphi)$) attractor of the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(\varphi)$, and $\Pi_2\mathcal{A} = \mathcal{H}(\varphi)$ $(\Pi_2(u, g) = g)$. Finally, by Corollary 2,

$$\mathcal{A}_\Sigma = \bigcup_{g \in \mathcal{H}(\varphi)} \mathcal{K}_g(0)\,, \qquad \Sigma = \mathcal{H}(\varphi)\,,$$

where $\mathcal{K}_g$ is the kernel of the process $\{U_g(t, \tau)\}$.

b) Non-autonomous reaction-diffusion system with a.p. terms.
Consider the following system:

$$\partial_t u = a\Delta u - f(u, t) + \varphi(x, t)\,, \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n\,, \tag{3.3}$$

(or with the boundary conditions $\partial u/\partial\nu|_{\partial\Omega} = 0$), where $a = \{a_{ij}\}_{i=1,\dots,N}^{j=1,\dots,N}$ is $N \times N$–matrix with a positive symmetric part $a + a^* \geq \beta^2 I$, $\beta > 0$, $u = u(x, t)$, $u = (u^1, \dots, u^N)$, $\varphi = (\varphi^1, \dots, \varphi^N)$, $f = (f^1, \dots, f^N)$. It is assumed that $\varphi(\cdot\,, t)$ is a bounded continuous function of $t \in \mathbb{R}$ with values in $H = (L_2(\Omega))^N$,

$\varphi(\cdot, t) \in L_\infty\big(\mathbb{R}, \big(L_2(\Omega)\big)^N\big)$. Also let $f, f'_{u^j} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^N)$, $(j = 1, \dots, N)$, and let the following conditions hold:

$$\gamma_2|u|^p - C_2 \le \big(f(u, t), u\big) \le \gamma_1|u|^p + C_1, \quad \gamma_i > 0, p \ge 2, \tag{3.4}$$

$$\big(f'_u(u, t)v, v\big) \ge -C_3(v, v) \quad \forall v \in \mathbb{R}^N, \tag{3.5}$$

$$\big|f'_u(u, t)\big| \le C_4\big(|u|^{p-2} + 1\big), \tag{3.6}$$

for any $t \in \mathbb{R}$, $u \in \mathbb{R}^N$. Note that (3.6) implies:

$$\big|f(u, t)\big| \le C_5\big(|u|^{p-1} + 1\big). \tag{3.7}$$

We supplement the system (3.3) with the initial conditions:

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in H = \big(L_2(\Omega)\big)^N. \tag{3.8}$$

Problem (3.3), (3.8) has (for all u_τ) a unique solution $u(t)$ belonging to

$$C_b\big([\tau,\infty), H\big) \cap L_2\big((\tau, T), \big(H_1^0(\Omega)\big)^N\big) \cap L_p\big((\tau, T), \big(L_p(\Omega)\big)^N\big).$$

Thus, there exists the process $\{U_{\sigma_0}(t, \tau)\}$, acting in the space $H = \big(L_2(\Omega)\big)^N$ and corresponding to problem (3.3), (3.8): $U_{\sigma_0}(t, \tau)\colon H \mapsto H$, $t \ge \tau$, $\tau \in \mathbb{R}$. System (3.3) has the time symbol $\sigma_0(t) = \big(f(u, t), \varphi(x, t)\big)$. Suppose that $\sigma_0(t)$ is a.p. function with values in the complete metric space $\mathcal{M}$ determined below. Let $\varphi(x, t)$ be an a.p. function with values in H, and $\mathcal{H}(\varphi)$ be the hull of this function. Let us define the Banach space $\mathcal{M}_1$ of functions $\psi(u) = \big(\psi^1(u), \dots, \psi^N(u)\big)$, $u = (u^1, \dots, u^N) \in \mathbb{R}^N$, with the following weighted norm:

$$\|\psi\|_{\mathcal{M}_1} = \sup_{u \in \mathbb{R}^N} \left(\frac{|\psi(u)|}{1 + |u|^{p-1}} + \frac{|\psi'_u(u)|}{1 + |u|^{p-2}}\right). \tag{3.9}$$

We assume that $f(u, t)$ is a.p. function with values in $\mathcal{M}_1$, $f(\cdot, t) \in C_b(\mathbb{R}, \mathcal{M}_1)$. Then the symbol $\sigma_0(t) = \big(f(u, t), \varphi(x, t)\big)$ is a.p. function with values in $\mathcal{M} = \mathcal{M}_1 \times H$. Consider the symbol space $\Sigma = \mathcal{H}(\sigma_0) = \mathcal{H}\big((f, \varphi)\big)$. Now consider the family of systems (3.3), (3.8) where $\sigma_0 = (f, \varphi)$ is replaced by any symbol $\sigma = (h, g) \in \mathcal{H}(\sigma_0) = \Sigma$. Obviously, if functions $f(u, t)$ and $\varphi(x, t)$ satisfy the conditions (3.4), (3.5), (3.6), then any pair $\big(h(u, t), g(x, t)\big)$ from Σ satisfy these conditions with the same constants C_i and γ_i. This follows directly from (3.6), (3.7) and the norm (3.9). Thus, the problem (3.3), (3.8) is correct and it generates the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in H. The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is uniformly compact, and $(H \times \Sigma, H)$–continuous (see [3], [4]). Thus, all the requirements of Theorem 2 are fulfilled for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, corresponding to problem (3.3) and, therefore,

i) the semigroup $\{S(t)\}$ corresponding to this family of processes has the compact attractor $\mathcal{A}$, $\mathcal{A} \subset\subset H \times \Sigma$;

ii) $\Pi_1 \mathcal{A} = \mathcal{A}_1 = \mathcal{A}_\Sigma$ is the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$;

iii) $\mathcal{A}_1 = \mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0)$, where $\mathcal{K}_\sigma$ is the kernel of $\{U_\sigma(t,\tau)\}$;

iv) for any $\sigma \in \Sigma$ the kernel $\mathcal{K}_\sigma$ is nonempty.

c) Non-autonomous nonlinear dissipative hyperbolic equation with a.p. terms.

The hyperbolic equations of the form

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u,t) + \varphi(x,t), \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \subset\subset \mathbb{R}^3, \tag{3.10}$$

where $\gamma > 0$, are considered. We assume that $f(u,t) \in C^2(\mathbb{R} \times \mathbb{R})$ and the following conditions hold:

$$F(u,t) = \int_0^u f(v,t)\,dv, \quad F(u,t) \ge -mu^2 - C_m, \tag{3.11}$$

$$f(u,t)u - cF(u,t) + mu^2 \ge -C_m, \tag{3.12}$$

where $m > 0$, $c > 0$, m is sufficiently small,

$$|f'_u(u,t)| \le C(1+|u|)^\rho, \quad |f'_t(u,t)| \le C(1+|u|)^{\rho+1}, \tag{3.13}$$

$$F'_t(t,u) \le \delta^2 F(t,u) + C, \tag{3.14}$$

where δ is sufficiently small, $\forall (t,u) \in \mathbb{R} \times \mathbb{R}$.

The case $\rho < 2$ for the autonomous equation (3.10) has been studied in [9], [7] and in works of other authors. The case $\rho = 2$ has been considered in [2], and others. We shall discuss here the case $\rho < 2$. We assume that $\varphi \in C_b(\mathbb{R}, L_2(\Omega))$. The initial conditions are posed at $t = \tau$:

$$u|_{t=\tau} = u_\tau(x), \quad \partial_t u|_{t=\tau} = p_\tau(x), \qquad \tau \in \mathbb{R}. \tag{3.15}$$

We shall write $y(t) = \big(u(t),\, \partial_t u(t)\big) = \big(u(t),\, p(t)\big)$, $y_\tau = (u_\tau, p_\tau) = y(\tau)$ for brevity. By E we denote the space of vector functions $y(x) = \big(u(x),\, p(x)\big)$ with finite energy norm $\|y\|_E^2 = \|u\|_1^2 + \|p\|^2$, $E = H_0^1(\Omega) \times L_2(\Omega)$, $y(t) \in E$, $t \geq \tau$. The unique solvability of problem (3.10), (3.15) in the energetic space E and properties of its solutions are established similarly to the autonomous case (see [12], [2], [14]). If $y_\tau \in E$ then the problem (3.10), (3.15) has a unique solution $y(t) \in C_b\big([\tau, +\infty), E\big)$. This implies that the process $\{U_{\sigma_0}(t, \tau)\}$ given by the identity $U_{\sigma_0}(t, \tau)y_\tau = y(t)$ is defined in E, where $\sigma_0(t) = \big(f(u, t),\, \varphi(x, t)\big)$ is the symbol of the equation (3.10). As usually, we assume that $\sigma_0(t)$ is a.p. function with values in a suitable metric space $\mathcal{M}$. To define $\mathcal{M}$ consider the Banach space $\mathcal{M}_2$ of functions $\big(\psi(u), \psi_1(u)\big)$, $u \in \mathbb{R}$, with the following weighted Banach norm:

$$\big\|\big(\psi(u),\psi_1(u)\big)\big\|_{\mathcal{M}_2} = \sup_{u\in\mathbb{R}} \left\{ \frac{|\psi(u)| + |\psi_1(u)|}{|u|^{\rho+1} + 1} + \frac{|\psi_u(u)|}{|u|^{\rho} + 1} \right\}.$$

We suppose that $\big(f(\cdot\,, t),\, f_t'(\cdot\,, t)\big)$ is an a.p. function of t with values in $\mathcal{M}_2$. We suppose also that $\varphi(\cdot\,, t)$ is a.p. function with values in $L_2(\Omega)$. Then $\sigma_0(t) = \big(f(u, t),\, \varphi(x, t)\big)$ is a.p. function with values in $\mathcal{M}_2 \times L_2(\Omega)$. Let $\mathcal{H}(\sigma_0)$ be the hull of function σ_0. Consider the symbol space $\Sigma = \mathcal{H}(\sigma_0)$. The problem (3.10), (3.15) with a symbol $\sigma(t) = \big(h(\cdot\,, t),\, g(\cdot\,, t)\big) \in \Sigma$ has a unique solution in the energetic space for any $\sigma \in \Sigma$, because the function $h(u, t)$ satisfies the conditions (3.11)–(3.14) with the same constants and $g(\cdot\,, t) \in C_b\big(\mathbb{R}, L_2(\Omega)\big)$. Hence, the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, acting in E, is defined, $U(t, \tau)\colon E \mapsto E$, $t \geq \tau$, $\tau \in \mathbb{R}$. The family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, corresponding to (3.10), (3.15) is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact and $(H \times \Sigma, H)$–continuous. The proof is given in [3] and [4]. It is analogous to that of given in [7] for autonomous case. A new point is the condition (3.14). It follows that for the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, and for the corresponding semigroup $\{S(t)\}$ Theorem 2 and Corollary 2 are applicable.

4. Hausdorff dimension estimates for attractors of non-autonomous dynamical systems with q.p. symbols

Consider Example 2. The symbol space of the system can be identify with k-dimensional torus $\mathbb{T}^k$. The system is:

$$\partial_t u = A(u, \alpha t + \omega_0), \quad \omega_0 \in \mathbb{T}^k, \ u|_{t=\tau} = u_\tau, \ u_\tau \in H, \ t \geq \tau, \ \tau \in \mathbb{R}, \qquad (4.1)$$

Here $A(u, \omega)$ is the family of nonlinear operators depending on $\omega \in \boldsymbol{T}^k$ with the domain H_1 and with values in H_0, where $H_1 \subseteq H \subseteq H_0$ are Hilbert spaces. Operators $A(u, \omega)$ are 2π-periodic with respect to each ω_j: $A(u, \omega_1, \ldots, \omega_j + 2\pi, \ldots, \omega_k) = A(u, \omega_1, \ldots, \omega_j, \ldots, \omega_k)$, $j = 1, \ldots, k$. It is assumed that the problem (4.1) is well posed and, consequently, there exists a unique solution $u(t) \in H$, $t \geq \tau$ (in a suitable functional space) for any symbol $\omega_0 \in \boldsymbol{T}^k$ and arbitrary $\tau \in \mathbb{R}$, $u_\tau \in H$. Hence, a family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \boldsymbol{T}^k$, corresponds to (4.1). It was proved in sec.2 that this family generates the semigroup $\{S(t)\}$, $S(t)\colon H \times \boldsymbol{T}^k \mapsto H \times \boldsymbol{T}^k$ acting by the formula (2.6). Evidently the semigroup $\{S(t)\}$ can be constructed using the following autonomous dynamical system:

$$\partial_t u = A(u, \omega)\,, \ \partial_t \omega = \alpha, \ u|_{t=0} = u_0\,, \quad \omega|_{t=0} = \omega_0\,, \ u_0 \in H, \ \omega_0 \in \boldsymbol{T}^k.$$

We assume that the family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \boldsymbol{T}^k$, is uniformly (w.r.t. $\omega_0 \in \boldsymbol{T}^k$) asymptotically compact and $(H \times \boldsymbol{T}^k, H)$–continuous. Therefore, by Theorem 2, the semigroup $\{S(t)\}$ possesses the compact (in $H \times \boldsymbol{T}^k$) attractor $\mathcal{A}$. The projection $\mathcal{A}_1$ of $\mathcal{A}$ onto H, $\mathcal{A}_1 = \Pi_1 \mathcal{A} = \mathcal{A}_{\boldsymbol{T}^k}$, is the uniform (w.r.t. $\omega_0 \in \boldsymbol{T}^k$) attractor of the family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \boldsymbol{T}^k$. Obviously

$$\dim \mathcal{A}_{\boldsymbol{T}^k} \leq \dim \mathcal{A}\,,$$

where $\dim \mathcal{A}_{\boldsymbol{T}^k}$ is the Hausdorff dimension in H of the uniform attractor $\mathcal{A}_{\boldsymbol{T}^k}$ and $\dim \mathcal{A}$ is the Hausdorff dimension in $H \times \boldsymbol{T}^k$ of the attractor $\mathcal{A}$. Therefore, in order to estimate $\dim \mathcal{A}_{\boldsymbol{T}^k}$, it is sufficient to get an upper bound for $\dim \mathcal{A}$.

a) 2D Navier–Stokes equations with q.p. external force.

We consider the family of Navier–Stokes systems (3.1) where $\varphi(x, t) = \Phi(x, \alpha t + \omega_0)$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $\omega_0 \in \boldsymbol{T}^k$, $\Phi(x, \omega) \in C(\boldsymbol{T}^k, H)$, $\Phi_{\omega_j}(x, \omega) \in C(\boldsymbol{T}^k, H)$, $\omega = (\omega_1, \ldots, \omega_k)$, $\Phi(x, \omega_1, \ldots, \omega_k)$ is a 2π-periodic function with respect to each $\omega_j (j = 1, \ldots, k)$. The following upper bound is valid for the Hausdorff dimension of the uniform attractor $\mathcal{A}_{\boldsymbol{T}^k}$ of the corresponding family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \boldsymbol{T}^k$:

$$\dim \mathcal{A}_{\boldsymbol{T}^k} \leq k + C_2 \Big(\frac{k}{\nu^2}\Big)^{1/3} + C_3 \Big(\frac{1}{\nu^2}\Big) \tag{4.2}$$

where C_2 and C_3 depend only on $|\Omega|$, g and g_{-1},

$$g = \sup_{\omega \in \boldsymbol{T}^k} \big\|\Phi'_\omega(\cdot\,, \omega)\big\| = \sup_{\omega \in \boldsymbol{T}^k} \Big(\sum_{j=1}^{k} \big\|\Phi'_{\omega_j}(\cdot\,, \omega)\big\|^2\Big)^{1/2},$$

$$g_{-1} = \sup_{\omega \in \boldsymbol{T}^k} \big\|G(\cdot\,, \omega)\big\|_{-1}\,.$$

Remark 1. In the autonomous case $k = 0$ the estimate (4.2) becomes the well-known upper bound c/ν^2 for the Hausdorff dimension of the attractor of the autonomous Navier–Stokes system ([5], [14]).

Remark 2. It is easy to construct an example of Navier–Stokes system with quasi-periodic external force $\varphi(x, t)$ such that $\dim \mathcal{A}_1 \geq k$. This implies that $\dim \mathcal{A}_1$ can increase when k is growing, while the Reynolds numbers remains bounded. Another examples have shown that $\dim \mathcal{A}_1$ can be infinite for an almost periodic external force $\varphi(x, t)$ having an infinite series of rationally independent frequencies.

b) Reaction-diffusion system with q.p. terms.
Consider the particular case of the system (3.3), where $f(u, t) = F(u, \omega_0 + \alpha t)$ and $\varphi(x, t) = \Phi(u, \omega_0 + \alpha t)$. Here $F(u, \omega)$ and $\Phi(u, \omega_0)$ are 2π-periodic with respect to each argument ω_j, $j = 1, \dots, k$, and $F, F'_u, F'_{\omega_j} \in C(\mathbb{R}^N \times \mathcal{T}^k, \mathbb{R}^N)$, $\Phi, \Phi'_{\omega_j} \in C\big(\mathcal{T}^k, (L_2(\Omega))^N\big)$. Let the conditions (3.4)–(3.7) be valid with f replaced by F and $t \in \mathbb{R}$ by $\omega \in \mathcal{T}^k$. Let also (for simplicity) $p < 2n/(n-2)$ for $n \geq 3$ and p is arbitrary positive for $n = 2$. We require also the following inequalities:

$$
\begin{aligned}
&\big|F'_\omega(u, \omega)\big| \leq C\big(1 + |u|^{(n+2)/(n-2)}\big) \quad (n > 2), \\
&\big|F(u+z, \omega+\mu) - F(u, \omega) - F'_u(u, \omega)z - F'_\omega(u, \omega)\mu\big| \leq \\
&\qquad \leq C\big(|u| + |z| + 1\big)^{p_1}\big(|z|^{1+\delta} + |\mu|^{1+\delta}\big), \\
&\big|\Phi(x, \omega+\mu) - \Phi(x, \omega) - \Phi'_\omega(x, \omega)\mu\big| \leq \psi(x)|\mu|^{1+\delta},
\end{aligned}
$$

$p_1 < 4/(n-2)$, $\forall u, z \in \mathbb{R}^N$, $\omega \in \mathcal{T}^k$, $\mu \in \mathbb{R}^k$, and $x \in \bar{\Omega}$. Here $\delta > 0$, δ is sufficiently small and $\psi \in L_2(\Omega)$. These conditions guaranties the quasidifferentiability of the semigroup $\{S(t)\}$ on the attractor $\mathcal{A}$ in $\big(L_2(\Omega)\big)^N \times \mathcal{T}^k$.

As we already know, the family of Cauchy problems (3.3) generates the family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \mathcal{T}^k$,acting in the space $\big(L_2(\Omega)\big)^N$. The corresponding semigroup $\{S(t)\}$ possesses the compact attractor $\mathcal{A} \subset\subset \big(L_2(\Omega)\big)^N \times \mathcal{T}^k$. The following estimate holds:

$$\dim \mathcal{A}_{\mathcal{T}^k} \leq \dim \mathcal{A} \leq k + C_2 k^{n/(n+2)} + C_3 . \tag{4.3}$$

where C_2 and C_3 do not depend on k.

Remark 3. Examples show that the main terms k in the estimates (4.2) and (4.3) are exact, under the condition that all other parameters controlling the norms of terms are bounded.

c) Nonlinear hyperbolic equation with q.p. in time terms.

Consider the particular case of the problem (3.10) with q.p. in time functions $f(u, t) = f_0(u, \omega_0 + \alpha t)$ and $\varphi(x, t) = \varphi_0(x, \omega_0 + \alpha t)$, $\alpha = (\alpha_1, \dots, \alpha_k)$ are rationally independent real numbers, $f_0(u, \omega)$ and $\varphi_0(x, \omega)$ are 2π-periodic with respect to each ω_j, $j = 1, \dots, k$, $\omega = (\omega_1, \dots, \omega_k)$. We assume that $f_0(u, \omega) \in C^2(\mathbb{R} \times \mathbb{T}^k, \mathbb{R})$, $\varphi_0(x, \omega) \in C^1(\mathbb{T}^k, L_2(\Omega))$, and $f_0(u, \omega)$ satisfies the conditions (3.11) - (3.14) with slight modifications. Besides, for $u, u^1 \in \mathbb{R}$, $\omega, \omega^1 \in \mathbb{T}^k$, we suppose that:

$$\left|f'_{0u}(u, \omega) - f'_{0u}(u^1, \omega^1)\right| \le C\left(|u|^{2-\delta} + |u^1|^{2-\delta} + 1\right)\left(|u - u^1|^\delta + |\omega - \omega^1|^\delta\right),$$

$$\left|f'_{0\omega}(u, \omega) - f'_{0\omega}(u^1, \omega^1)\right| \le C\left(|u|^{3-\delta} + |u^1|^{3-\delta} + 1\right)\left(|u - u^1|^\delta + |\omega - \omega^1|^\delta\right),$$

$$\left|\varphi_0(x, \omega + \mu) - \varphi_0(x, \omega) - (\varphi'_{0\omega}(x, \omega), \mu)\right| \le C|\psi(x)||\mu|^{1+\delta},$$

$\mu = (\mu_1, \dots, \mu_k) \in \mathbb{R}^k$, $0 < \delta \le 1$, $\psi(x) \in L_2(\Omega)$. The family of equations (3.10) generates the family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \mathbb{T}^k$, acting in the energy space E. Using the usual scheme, one can construct a semigroup $\{S(t)\}$ acting in the extended phase-space $\mathcal{E} = E \times \mathbb{T}^k$, $S(t): \mathcal{E} \mapsto \mathcal{E}$. The family of processes $\{U_{\omega_0}(t, \tau)\}$, $\omega_0 \in \mathbb{T}^k$, possesses the compact uniform (w.r.t. $\omega_0 \in \mathbb{T}^k$) attractor $\mathcal{A}_1 \subset\subset E$ consisting of all bounded in E complete trajectories $y(t) = \left(u(t), \partial_t u(t)\right)$, $t \in \mathbb{R}$, of the equation (3.10) with an arbitrary symbol $\omega_0 \in \mathbb{T}^k$ (see [3], [4]).

We can show that the uniform attractor $\mathcal{A}_1$ of the equation (3.10) with q.p. symbol has finite Hausdorff dimension. The estimate for $\dim \mathcal{A}_1$ depending on the number k of rationally independent frequencies $(\alpha_1, \dots, \alpha_k)$ of the q.p. symbol:

$$\dim \mathcal{A}_1 \le C_1 k + C_2 k^{1/3} + C_3 . \tag{4.4}$$

R e m a r k 4. For the hyperbolic equation the main term in the estimate (4.4) equals to $C_1 k$ ($C_1 > 1$) while in (4.2) and (4.3) $C_1 = 1$.

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