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ASYMPTOTIC BEHAVIOR FOR SEMILINEAR DAMPED WAVE EQUATIONS ON $\mathbb{R}^N$

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ABSTRACT. Large time asymptotic behavior of solutions to the problem

$$u_{tt} + du_t - \Delta u + f(x,u) = 0, \ u = u(x,t), \quad x \in \mathbb{R}^N, \ t > 0, \ d > 0,$$

is considered with respect to various structural properties of the nonlinearity f.

We shall discuss the long time behavior of solutions of the problem

$$u_{tt} + du_t - \Delta u + f(x,u) = 0\,, \ u = u(x,t)\,, \quad x \in \mathbb{R}^N, \ t > 0, \ d > 0\,, \tag{E}$$

$$(u, u_t)(\cdot\,, 0) \in X = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\,. \tag{I}$$

Two rather different situations are considered :

(**A**) If the nonlinearity f is *coercive for large* x, the dynamics is asymptotically compact like for the corresponding problem on a bounded spatial domain. More specifically, we report the following result :

PROPOSITION 1. [2, Theorem 1]. Let $N = 3$. Under the hypotheses

$$f \in C^2(\mathbb{R}^4), \ f(\cdot\,,0) \in H^1(\mathbb{R}^3), \ |f_z(x,0)| \le C \qquad \text{for all } x \in \mathbb{R}^3, \tag{1}$$

$$|f_{zz}(x,z)| \le C(1+|z|) \qquad \text{for all } x, z\,, \tag{2}$$

$$\liminf_{|z|\to\infty} \frac{f(x,z)}{z} \ge 0 \qquad \text{uniformly in } x \in \mathbb{R}^3, \tag{3}$$

$$\big(f(x,z) - f(x,0)\big)z \ge Cz^2, \qquad C > 0\,, \quad \text{for all } x \text{ large}\,, \tag{4}$$

there exists a unique global attractor $\mathcal{A}$ of the semigroup

$$S_t \colon (u, u_t)(0) \to (u, u_t)(t)$$

on X, i.e.,

$$\mathcal{A} \subset X \quad \text{is compact}\,, \tag{5}$$

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$$S_t(\mathcal{A}) = \mathcal{A} \quad \text{for all } t \geq 0\,, \tag{6}$$

$$\operatorname{dist}\big(S_t(\mathcal{B}), \mathcal{A}\big) \to 0 \quad \text{as} \quad t \to \infty \tag{7}$$

$$\text{for any bounded} \quad \mathcal{B} \subset X\,,$$

where

$$\operatorname{dist}(\mathcal{C}, \mathcal{D}) \equiv \sup_{c \in \mathcal{C}} \inf_{d \in \mathcal{D}} \|c - d\|_X\,.$$

R e m a r k. Though the result is formulated for $N = 3$, there are no essential difficulties to prove the same for a general N the growth condition (2) being modified properly.

(**B**) For a general *noncoercive* f, i.e., when $F(z) = \int_0^z f(s)\,ds$ is allowed to be negative for certain values of the argument z, the dynamics exhibits truly infinite-dimensional character though some compactness results are still possible. We assume that $f = f(u)$ along with the following hypotheses

$$f \in C^1(\mathbb{R}),\ f(0) = 0,\ f'(0) = a > 0\,, \tag{8}$$

$$f(u)u \geq -Cu^2 \quad \text{for all } u \in \mathbb{R}\,, \tag{9}$$

$$|f'(u)| \leq C(1 + |z|^q) \quad \text{with} \quad 2(q+1) < \frac{2N}{N-2} \tag{10}$$

$$\text{if } N > 2\,,\ q \text{ arbitrary finite otherwise}.$$

According to the recent state of affairs, the main features of the problem may be characterized as follows :

1. If $F(w) = \int_0^w f(s)\,ds < 0$ for certain w and $N \geq 3$, then there is a sequence $\{\bar{u}_n\}$ of finite energy stationary states, i.e., $\bar{u}_n$ solve

$$-\Delta v + f(v) = 0, \quad v \in H^1(\mathbb{R}^N)\,, \tag{11}$$

such that

$$T(\bar{u}_n) \to \infty \quad \text{as} \quad n \to \infty\,, \quad T(v) = \frac{1}{2}\Big(\int |\nabla v|^2 + 2F(v)\,dx\Big)$$

(see B e r e s t y c k i–L i o n s [1]).

2. The zero solution $\bar{u}_0 \equiv 0$ is the only stable steady state in X (see K e l l e r [5]).

3. The solution semigroup $\{S_t\}$ is not dissipative in X, in other words, the damping term du_t is not strong enough to ensure boundedness of the trajectories in X ([4, Corollary 5])

In this case, we claim the following :

PROPOSITION 2. [4, Theorem 1]. *Under the above hypotheses, let*

$$u \in C(\mathbb{R}^+, H^1), \quad u_t \in C(\mathbb{R}^+, L^2)$$

be a (weak) solution to (E) such that there is a sequence $\{t_n\}$, $t_n \to \infty$,

$$\|(u, u_t)(t_n)\|_X \le C < \infty . \tag{12}$$

Then (passing to a subsequence if necessary) we have

$$\|(u, u_t)(t_n) - \sum_{j=1}^{k} (\bar{u}_j(\,\cdot + x_j^n), 0)\|_X \to 0 \quad \text{as} \quad n \to \infty , \tag{13}$$

where k is a finite integer, $\bar{u}_j$, $j = 1, \dots, k$, are (not necessarily distinct) solutions of (11) and x_j^n, $x_i^n \in \mathbb{R}^N$,

$$\operatorname{dist}(x_j^n, x_i^n) \to \infty \quad \text{for} \quad i \ne j,\ n \to \infty . \tag{14}$$

Proposition 2 is proved by means of the concentration compactness theory due to L i o n s [6].

Finally, it may be shown that even in case (B) there is a chance to obtain compactness changing the phase space appropriately. In addition to the above hypotheses, we shall assume

$$\liminf_{|z|\to\infty} \frac{f(z)}{z} \ge b > 0 , \qquad f'(z) \ge -C \quad \text{for all } z . \tag{15}$$

Next, we introduce the norm

$$\|v\|^2_{L^2_B} = \sup_{y\in\mathbb{R}^N} \int\limits_{|x-y|\le 1} v^2 \, dx \tag{16}$$

along with the corresponding space L^2_B defined as a completion of the set of all smooth and bounded functions on $\mathbb{R}^N$ with respect to $\|\ \|_{L^2_B}$. In a similar way, the space H^1_B is defined by means of the norm

$$\|v\|^2_{H^1_B} = \|\,|\nabla v|\,\|^2_{L^2_B} + \|v\|^2_{L^2_B} . \tag{17}$$

Finally, we write

$$X_B = H^1_B \times L^2_B . \tag{18}$$

It may be shown (see [3, Section 2]) that the Cauchy problem for (E) is well posed on X_B, and that the solution operator $\{S_t\}$ forms a group of locally Lipschitz continuous mappings on X_B .

Our final result reads as follows :

PROPOSITION 3. [3, Theorem 1]

There is a set $\mathcal{A} \subset X_B$ *enjoying the following properties :*

(A1) $\mathcal{A}$ *attracts bounded sets in* X_B *, i.e., for any* $\mathcal{B}(u, u_t) \subset X_B$ *bounded, we have*

$$\operatorname{dist}\big(S_t(\mathcal{B}), \mathcal{A}\big) \to 0 \quad \textit{as} \quad t \to \infty ,$$

(A2) $\mathcal{A}$ *is time invariant, i.e.,*

$$S_t(\mathcal{A}) = \mathcal{A} \quad \textit{for all} \quad t \geq 0 ,$$

(A3) $\mathcal{A}$ *is locally compact in the sense that* $\mathcal{A}$ *is bounded in* X_B *and compact in* X_{loc}*, where*

$$X_{loc} = H^1_{\mathrm{loc}}(\mathbb{R}^N) \times L^2_{\mathrm{loc}}(\mathbb{R}^N) .$$

Remark. It is clear that $\mathcal{A}$ is uniquely determined by the conditions (A1) – (A3). Moreover, any set satisfying (A1), (A2) contains $\mathcal{A}$. This justifies the denomination global attractor for $\mathcal{A}$.

The proof of Proposition 3 does not use the conclusion of Proposition 2. The main idea is to work in weighted Sobolev spaces with weights polynomially decreasing for large values of $|x|$.

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