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APPLICATIONS OF REPRESENTATION THEOREMS FOR BIMEASURES

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ABSTRACT. Theorems on representing bimeasures, i.e., separately σ-additive complex functions on the Cartesian product of two algebras of sets, in terms of spectral measures and positive-definite bimeasures are applied to obtain a Carathéodory–Hahn type extension theorem for bimeasures and a decomposition theorem involving positive-definite kernels on Cartesian squares of locally compact abelian groups.

1. Introduction and notation

Throughout this note, S_i will be a nonempty set, Σ_i an algebra (field) of subsets of S_i for $i = 1, 2$, and $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ a bounded separately (finitely) additive function. In case β is separately σ-additive (i.e., $\beta(X, \cdot)$ and $\beta(\cdot, Y)$ are countably additive for all $X \in \Sigma_1$, $Y \in \Sigma_2$), β will be called a (complex) *bimeasure*. Section 2 contains an exposition of certain representation theorems for bimeasures proved in [10]. In Sections 3 and 4 we apply some of these results and prove a Carathéodory–Hahn type extension theorem and a theorem on expressing certain functions on $G \times G$ for a locally compact abelian group G as linear combinations of positive-definite kernels.

We now explain the essential concepts and notation needed in the sequel. The characteristic function of a set $A \subset S_i$ is denoted by χ_A. We let $\mathcal{F}_i$ be the space of finite linear combinations of χ_A for $A \in \Sigma_i$, and write $\mathcal{C}_i$ for the closure of $\mathcal{F}_i$ in the space of bounded complex functions on S_i equipped with the supremum norm.

Since β is bounded and separately additive, its semivariation (in the sense of [8, p. 120]) is finite (see [8, p. 121]), and it is easily seen that there is a unique bounded bilinear function $B : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathbb{C}$ satisfying

$$B(\chi_X, \chi_Y) = \beta(X, Y)$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. We write

$$B(f, g) = \int (f, g)\, d\beta$$

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for $f \in \mathcal{C}_1$, $g \in \mathcal{C}_2$, and call this number the *integral* of the pair (f,g) with respect to β; this is consistent with [8], see [8, p. 126]. For the basic theory of bimeasures (defined on products of σ-algebras), one may consult, e.g., [1] and [8].

For any set T, a function $F : T \times T \to \mathbb{C}$ is called a *positive-definite kernel,* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \bar{c}_j F(t_i, t_j) \geq 0$$

for all finite sequences $t_1, \ldots, t_n \in T$, $c_1, \ldots, c_n \in \mathbb{C}$. In the case of only one algebra Σ of subsets of a nonempty set S, in particular, a bounded-separately finitely additive function $\beta : \Sigma \times \Sigma \to \mathbb{C}$ is said to be *positive-definite,* if it is a positive-definite kernel. This condition is easily seen to be equivalent to requiring that

$$\int (f, \bar{f})\, d\beta \geq 0$$

for every f in the space $\mathcal{C}$ of functions that can be expressed as uniform limits of linear combinations of characteristic functions of sets in Σ.

All vector spaces will be complex. For any Hilbert space H, $(\cdot|\cdot)$ denotes its inner product and $L(H)$ the space of bounded linear operators on H. If H is a Hilbert space, a mapping $E : \Sigma_i \to L(H)$ is called a *finitely additive spectral measure* if $E(S_i) = I(= \mathrm{id}_H)$, $E(X) = E(X)^* = E(X)^2$ for all $X \in \Sigma_i$, and $E(X \cup Y) = E(X) + E(Y)$ for disjoint $X, Y \in \Sigma_i$. We call such an E *strongly σ-additive* if E is countably additive with respect to the strong (or, equivalently, to the weak) operator topology.

2. Representation theorems

The proofs of the results in this section may be found in [10]. Some key techniques used there are based on the Grothendieck inequality [4] and the Yosida–Hewitt decomposition [11].

THEOREM 2.1. *There is a Hilbert space H with vectors ξ, $\eta \in H$, and finitely additive spectral measures $E_i : \Sigma_i \to L(H)$ such that*

$$\beta(X, Y) = \big(E_1(X)E_2(Y)\xi|\eta\big)$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. If β is separately σ-additive, then the E_i can be taken to be strongly σ-additive.

The rest of the theorems in this section deal with only one algebra of sets.

THEOREM 2.2. *Let S be a nonempty set and Σ an algebra of subsets of S. Let $\beta : \Sigma \times \Sigma \to \mathbb{C}$ be a bounded separately additive function.*

(a) *There is a Hilbert space* H *with a finitely additive spectral measure* $E : \Sigma \to L(H)$, *a bounded linear operator* $T : H \to H$ *and a vector* $\xi \in H$ *such that*

$$\beta(X,Y) = \big(E(X)TE(Y)\xi|\xi\big)$$

for all $X, Y \in \Sigma$.

(b) *If* β *is separately* σ*-additive, then* E *in* (a) *can be taken to be strongly* σ*-additive.*

THEOREM 2.3. *Let* S *be a nonempty set and* Σ *an algebra of subsets of* S. *Let* $\beta : \Sigma \times \Sigma \to \mathbb{C}$ *be a bounded separately additive function.*

(a) *The following conditions are equivalent:*

(i) β *is positive definite;*

(ii) *there is a Hilbert space* H *with a finitely additive spectral measure* $E\colon \Sigma \to L(H)$, *an orthogonal projection* $P\colon H \to H$ *and a vector* $\xi \in H$ *such that*

$$\beta(X,Y) = \big(E(X)PE(Y)\xi|\xi\big)$$

for all X, $Y \in \Sigma$;

(iii) *there is a Hilbert space* H *with a finitely additive spectral measure* $E : \Sigma \to L(H)$, *a positive operator* $T \in L(H)$, *and a vector* $\xi \in H$ *such that*

$$\beta(X,Y) = \big(E(X)TE(Y)\xi|\xi\big)$$

for all X, $Y \in \Sigma$.

(b) *If* β *is separately* σ*-additive, then* E *in* (a) *can be taken to be strongly* σ*-additive.*

THEOREM 2.4. *Let* S *be a nonempty set and* Σ *an algebra of subsets of* S. *Let* $\beta\colon \Sigma \times \Sigma \to \mathbb{C}$ *be a bounded separately additive function.*

(a) *There are four positive-definite bounded separately additive functions* $\beta_1, \dots, \beta_4$ *on* $\Sigma \times \Sigma$ *such that*

$$\beta = \beta_1 - \beta_2 + i(\beta_3 - \beta_4).$$

(b) *If* β *is separately* σ*-additive, then the* $\beta_1, \dots, \beta_4$ *in* (a) *can be taken to be separately* σ*-additive.*

3. A Carathéodory–Hahn type extension theorem

In [3] Dobrakov considers the problem of extending a separately σ-additive function defined on the Cartesian product of two rings of sets to a separately σ-additive function on the product of the generated σ-rings. We now show that in the case of algebras of sets such an extension always exists. As before, S_i is a nonempty set, and Σ_i is an algebra of subsets of S_i for $i = 1, 2$.

THEOREM 3.1. *Let $\tilde{\Sigma}_i$ be the σ-algebra generated by Σ for $i = 1, 2$. If $\beta\colon \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a separately σ-additive function, it has a unique separately σ-additive extension $\tilde{\beta} : \tilde{\Sigma}_1 \times \tilde{\Sigma}_2 \to \mathbb{C}$.*

Proof. Using Theorem 2.1 we find a Hilbert space H with vectors $\xi, \eta \in H$ and strongly σ-additive spectral measures $E_i\colon \Sigma_i \to L(H)$ such that

$$\beta(X, Y) = \big(E_1(X)E_2(Y)\xi|\eta\big)$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. It is a classical fact that E_i can be extended to a strongly σ-additive spectral measure $\tilde{E}_i : \tilde{\Sigma}_i \to L(H)$. (One way of proving this is to reduce it to the Carathéodory–Hahn extension theorem applied to the nonnegative set functions $\big(E_j(\cdot)\xi|\xi\big)$ for $\xi \in H$. An outline of the essential steps in a somewhat specialized situation may be found e.g. in [5, pp. 72–73].) Defining $\tilde{\beta}(X, Y) = \big(\tilde{E}_1(X)E_2(Y)\xi|\eta\big)$ for $X \in \tilde{\Sigma}_1$, $Y \in \tilde{\Sigma}_2$ we get a separately σ-additive extension of β. The uniqueness of the extension is seen by applying twice the uniqueness of the usual Carathéodory–Hahn extension. □

4. Linear combinations of some positive-definite kernels

In this section, G is a locally compact abelian group, and Γ its dual group. We refer to [6] for terminology and use generally its notation. We write, in particular, the group operations additively and (x, γ) for the value of a continuous character $\gamma \in \Gamma$ at $x \in G$. Let $M(G)$ stand for the space of (bounded) regular complex Borel measures on G, and define $M(\Gamma)$ similarly. The space of finite linear combinations of point measures δ_x with $x \in G$ is denoted by $M_{dd}(G)$. If $\mu = \sum_{i=1}^{n} c_i \delta_{x_i} \in M_{dd}$, then for each $\gamma \in \Gamma$ we write

$$\hat{\mu}(\gamma) = \sum_{i=1}^{n} c_i \overline{(x_i, \gamma)}.$$

The function $\hat{\mu}$ so defined is the *Fourier-Stieltjes transform* of μ.

We let $B_2(G, G)$ denote the set of the separately continuous functions $f\colon G \times G \to \mathbb{C}$ satisfying

$$\Big|\sum_{i=1}^{m} \sum_{j=1}^{n} c_i d_j f(x_i, y_j) Bbig| \leq 1$$

whenever $\mu = \sum_{i=1}^{m} c_i \delta_{x_i} \in M_{dd}(G)$ and $\nu = \sum_{j=1}^{n} d_j \delta_{y_j} \in M_{dd}(G)$ satisfy

$$\sup_{\gamma \in \Gamma} \big|\hat{\mu}(\gamma)\big| \leq 1$$

and

$$\sup_{\gamma \in \Gamma} \big|\hat{\nu}(\gamma)\big| \leq 1.$$

It turns out that the functions in $B_2(G,G)$ are actually jointly continuous. For the proof of this fact and several characterizations of $B_2(G,G)$, cf. [7, p. 377]. We let $\mathcal{B}(\Gamma)$ denote the Borel σ-algebra of Γ and $C_0(\Gamma)$ the space of continuous complex functions on Γ vanishing at infinity, equipped with the supremum norm. Motivated by [7, Theorem 5.7], one might call $B_2(G,G)$ the space of Fourier transforms of bounded bilinear forms on $C_0(\Gamma)\times C_0(\Gamma)$ (or, in view of [8, Lemma 6.5] equivalently, of separately regular bimeasures on $\mathcal{B}(\Gamma)\times\mathcal{B}(\Gamma)$).

THEOREM 4.1. *If $F\in B_2(G,G)$, then there are four positive-definite kernels $F_1,\dots,F_4$ in $B_2(G,G)$ such that*

$$F = F_1 - F_2 + i(F_3 - F_4).$$

Proof. We identify as usual the dual of $C_0(\Gamma)$ with $M(\Gamma)$, and so each bounded Borel function $f\colon\Gamma\to\mathbb{C}$ determines an element $\tilde{f}$ of the bidual $C_0(\Gamma)^{**}$ of $C_0(\Gamma)$ via the formula $\langle\mu,\tilde{f}\rangle = \int_\Gamma f\,d\mu$, $\mu\in M(\Gamma)$. Whenever $B : C_0(\Gamma)\times C_0(\Gamma)\to\mathbb{C}$ is a bounded bilinear form, there is a unique separately weak*-continuous bilinear extension

$$\tilde{B} : C_0(\Gamma)^{**}\times C_0(\Gamma)^{**}\to\mathbb{C}.$$

(See [7], p. 366.) If $f,g\colon\Gamma\to\mathbb{C}$ are bounded Borel functions, then $\tilde{B}(\tilde{f},\tilde{g})$ coincides with the integral $\int(f,g)\,d\beta$ where $\beta\colon\mathcal{B}(\Gamma)\times\mathcal{B}(\Gamma)\to\mathbb{C}$ is the bimeasure defined by $\beta(X,Y)=\tilde{B}(\chi_X,\chi_Y)$ (see [8, p. 128]). As in [7], we use in this situation the notation $B(f,g)=\tilde{B}(\tilde{f},\tilde{g})$. From [7, Theorem 5.7] it follows that there is some bounded bilinear form $B_0 : C_0(\Gamma)\times C_0(\Gamma)\to\mathbb{C}$ such that

$$F(x,y)=\tilde{B}_0\big((x,\cdot),(-y,\cdot)\big)$$

for all x, $y\in G$. Let $\beta_0 : \mathcal{B}(\Gamma)\times\mathcal{B}(\Gamma)\to\mathbb{C}$ be the bimeasure corresponding to B_0 as in the above discussion. Using Theorem 2.4 we find four positive-definite bimeasures $\beta_1,\dots,\beta_4\colon\mathcal{B}(\Gamma)\times\mathcal{B}(\Gamma)\to\mathbb{C}$ such that

$$\beta_0=\beta_1-\beta_2+i(\beta_3-\beta_4).$$

We now define $B_j(f,g)=\int(f,g)\,d\beta_j$ for $f,g\in C_0(\Gamma)$, $j=1,\dots,4$. Each B_j is a bounded bilinear form satisfying $B_j(f,\overline{f})\ge 0$ for all $f\in C_0(\Gamma)$. Theorem 5.8 in [7, p. 378] shows that each function $(x,y)\mapsto F_j(x,y)=B_j\big((x,\cdot),(-y,\cdot)\big)$ on $G\times G$ is a positive-definite kernel, and by [7, Theorem 5.7] it belongs to $B_2(G,G)$. Clearly $F=F_1-F_2+i(F_3-F_4)$. □

REMARK 4.2. (a) It can be shown that in the above proof the bimeasures β_j may be taken to be separately regular. Theorem 2.4 does not directly yield this fact, and we do not need it. It is worth noting, however, that in the absence of separate regularity the formula $\beta_j(X,Y)=\tilde{B}_j(\tilde{\chi}_X,\tilde{\chi}_Y)$ for X, $Y\in\mathcal{B}(\Gamma)$ is not necessarily valid.

(b) If $B : C_0(\Gamma)\times C_0(\Gamma)\to\mathbb{C}$ is a bounded bilinear form, it is possible to find without (bi)measure theoretic considerations four bounded bilinear forms

$B_j : C_0(\Gamma) \times C_0(\Gamma) \to \mathbb{C}$ satisfying $B_j(f, \overline{f}) \geq 0$ for all $f \in C_0(\Gamma)$, such that $B = B_1 - B_2 + i(B_3 - B_4)$; even a much more general result follows from the heavy machinery of [2]: see page 157 and Corollaries 4.3 and 5.6. The point is that in view of the Grothendieck inequality every bounded bilinear form $B\colon C_0(\Gamma) \times C_0(\Gamma) \to \mathbb{C}$ is *completely bounded* in the sense of [2], and such bilinear forms (and more generally, operator-valued multilinear operators on products of not necessarily commutative C^*-algebras) have simple representation and decomposition properties. Fourier transforms of multilinear operators of this type on products of noncommutative group C^*-algebras have been discussed in [9], and it is clear that Theorem 4.1 has various generalizations to that setting. In keeping with the title of the present paper, we leave further discussion of that topic.

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