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DECOMPOSITIONS OF RIESZ SPACE–VALUED MEASURES ON ORTHOMODULAR POSETS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. We present a decomposition theorem showing that any positive finitely additive measure defined on an orthomodular poset attaining values in a Dedekind complete normed Riesz space can be expressed as a sum of two finitely additive measures, where the first one belongs to a given cone of measures, and the second one is singular with respect to the cone. As corollaries we obtain Yosida-Hewitt-type decompositions giving cones of σ-additive measures, completely additive measures, $\mathcal{P}$-regular measures, Lebesgue-type-decomposition, and Aarnes decomposition on inner product spaces.

1. Introduction

The classical decomposition theorems of Yosida–Hewitt [20] and Lebesgue [12] have received in last years attention of authors studying finitely additive measures on orthomodular posets which generalize algebras of sets [5, 18, 4, 6, 7]. In the present paper, we give a general decomposition theorem for Riesz space-valued, finitely additive measures defined on an orthomodular posets. Our method generalizes that one from [5] which is a natural refinement of the original arguments of Yosida and Hewitt [20]. We recall that it also reflects some reasonings of Rüttimann [18] for real-valued measures. As corollaries we obtain many familiar results from the papers [1, 20, 12, 18, 5, 6, 7]. Some comments concerning Gleason's theorem and decomposition of real-valued measures defined on splitting subspaces of inner product spaces are present in details.

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2. Orthomodular posets

An *orthomodular poset* is a partially ordered set L with an ordering $\leq$, the minimal and maximal elements 0 and 1, respectively, and an orthocomplementation $\perp : L \to L$ such that

(i) $a^{\perp\perp} = a$ for any $a \in L$;
(ii) $a \vee a^{\perp} = 1$ for any $a \in L$;
(iii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$;
(iv) if $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$;
(v) if $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ $(= a \vee (b \wedge a^{\perp}))$ (orthomodular law).

We recall that from the above axioms we have de Morgan laws

$$\Big(\bigvee_i a_i\Big)^{\perp} = \bigwedge_i a_i^{\perp} \quad \text{and} \quad \Big(\bigwedge_i a_i\Big)^{\perp} = \bigvee_i a_i^{\perp}$$

saying that if one side of an equality exists in L, so exists the second one, and both are equal. If in an orthomodular poset L the join of any sequence (any system) of mutually orthogonal elements exists, we say that L is a σ-*orthomodular poset* (a *complete orthomodular poset*). An *orthomodular lattice* is an orthomodular poset L such that, for any $a, b \in L$, $a \vee b$ exists in L (using de Morgan laws, $a \wedge b$ exists in L, too). A distributive orthomodular lattice is called a *Boolean algebra.* We recall that an orthomodular lattice L is a Boolean algebra iff for any pair $a, b \in L$ there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$.

We recall that orthomodular posets are intensively studied due to their intimate connection with mathematical foundations of quantum mechanics (for more details concerning orthomodular posets and lattices see, e.g. [13, 17]).

One of the most important cases of orthomodular lattices is the system of all closed subspaces, $L(H)$, of a real or complex Hilbert space H, with an inner product $(\cdot,\cdot)$. Here the partial ordering, $\leq$, is induced by the natural set-theoretic inclusion, and $M^{\perp} = \{x \in H\colon\ (x, y) = 0 \text{ for any } y \in M\}$. Then $L(H)$ is a complete orthomodular lattice, which is not a Boolean algebra, if $\dim H \neq 1$.

If S is an inner product space (not necessarily complete), denote by $E(S)$ the set of all *splitting subspaces* of S, i.e., the set of all $M \subseteq S$ such that $M + M^{\perp} = S$. Then $E(S)$ is an orthomodular poset which is not necessarily a σ-orthomodular poset. We recall that according to [8], S is complete if and only if $E(S)$ is a σ-orthomodular poset.

3. Riesz spaces

Let V be a real vector space with a partial ordering $\leq$ such that

(i) if $x, y \in V$, then $x \wedge y \in V$, $z \vee y \in V$;

(ii) if $x \leq y$, then $x + z \leq y + z$, for any $z \in V$;

(iii) if $x \leq y$, then $\alpha x \leq \alpha y$ for any $\alpha \in \mathbb{R}_+$,

then V is said to be a *Riesz space.* We define for any $x \in V$: $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x^+ + x^-$. The set $[x, y] := \{z \in V : x \leq z,\ z \leq y\}$, where $x, y \in V$, $x \leq y$, is called an *order interval.*

A non-void set D of V is said to be *directed downwards*, and we write $D\downarrow$, if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$. For a directed downwards set D, the set $D_x = \{y \in D : y \leq x\}$, where $x \in D$, is called a *section* of D determined by x; the system $\{D_x : x \in D\}$ is a filter base in V for a filter $\mathcal{F}(D)$ called a *filter of sections* on D. We say that a filter $\mathcal{F}$ on V is *order convergent* to a vector $x \in V$ if $\mathcal{F}$ contains a family of order intervals with intersection $\{x\}$.

A Riesz space V is called *Dedekind complete* if, for every non-void majorized subset B of V, $\bigvee B := \bigvee\{b : b \in B\}$ exists in V.

A norm $\|\cdot\|$ on a Riesz space V is said to be a *Riesz norm* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, and a pair $(V, \|\cdot\|)$ is called a *normed Riesz space*, if, moreover, $(V, \|\cdot\|)$ is complete, it is called a *Banach lattice.*

A normed Riesz space is said to have *order continuous norm* if every order convergent filter in V converges in norm to its order limit. We recall that any Banach lattice with order continuous norm is Dedekind complete.

The following are well-known examples of Dedekind complete normed Riesz spaces with order continuous norm:

(1) The n-dimensional vector space $\mathbb{R}^n$ with its canonical order and norm;

(2) $L_p(\Omega, \Sigma, \mu)$ with $1 < p < \infty$;

(3) $L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_L)$, where μ_L is the Lebesgue measure;

(4) Any reflexive Banach lattice.

4. Decomposition theorem

In the present part, we give the main result of the paper. The proof follows ideas developed in [5], and to be self-contained we present our proof in the full generality.

Throughout this paper by L we understand an orthomodular poset and by $(V, \|\cdot\|)$ a Dedekind complete normed Riesz space with order continuous norm.

Now let W be an arbitrary normed Riesz space. Define the following partial ordering $\leq_n$ on W^L: $\mu_1 \leq_n \mu_2$ if $\mu_1(a) \leq \mu_2(a)$ for every $a \in L$. We say that $\mu \in W^L$ is

(a) *finitely additive* if $\mu(a \vee b) = \mu(a) + \mu(b)$ whenever $a \perp b$;

(b) σ-*additive* if, for any sequence of mutually orthogonal elements $\{a_n\}_{n=1}^{\infty}$ of L for which $\bigvee_{n=1}^{\infty} a_n$ exists in L, we have $\mu(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \mu(a_n)$ (in the norm topology of W);

(c) *completely additive* if, for any system of mutually orthogonal elements $\{a_i\}_{i \in I}$ of L for which the join $\bigvee_{i \in I} a_i$ exists in L, we have that the family $\{\mu(a_i)\colon i \in I\}$ is summable in W and $\mu(\bigvee_{i \in I} a_i) = \sum_{i \in I} \mu(a_i)$ (in the norm topology of W).

We denote by $a(L,W)_+$, $\sigma a(L,W)_+$, and $ca(L,W)_+$ the sets of all positive finitely additive, σ-additive, and completely additive measures, respectively, from W^L. We recall that $ca(L,W)_+ \subseteq \sigma a(L,W)_+ \subseteq a(L,W)_+$, and $\mu(0) = 0$, if $\mu \in a(L,W)_+$.

Let $\mathcal{C}$ be a non-void subset of $a(L,W)_+$. We say that $\mathcal{C}$ is a *quasi cone* of $a(L,W)_+$ if the zero function on L belongs to $\mathcal{C}$, and $\mu_1 + \mu_2 \in \mathcal{C}$ whenever $\mu_1, \mu_2 \in \mathcal{C}$. $\mathcal{C}$ is *uniformly closed* if for a net $\{\mu_t\}$ in $\mathcal{C}$ and an element $\mu \in a(L,W)_+$ such that $\|\mu_t(c) - \mu(c)\| \to 0$ uniformly for $c \in L$, we have $\mu \in \mathcal{C}$. We say that an element $\mu \in a(L,W)_+$ is *singular* with respect to the quasi cone $\mathcal{C}$ if $\nu \leq_n \mu$ for some $\nu \in \mathcal{C}$ implies $\nu = 0$. We denote by $\mathcal{C}^\sharp$ the set of all elements of $a(L,W)_+$ which are singular with respect to $\mathcal{C}$.

It is evident that if $\{\mathcal{C}_i : i \in I\}$ is a system of uniformly closed quasi cones of $a(L,W)_+$, so is $\mathcal{C} = \bigcap_{i \in I} \mathcal{C}_i$, and $\mathcal{C}$ can be used for different kinds of decompositions.

THEOREM 4.1. *Let $\mathcal{C}$ be a uniformly closed quasi cone of $a(L,V)_+$. Then for any $\mu \in a(L,V)_+$ there exist two elements $\xi \in \mathcal{C}$ and $\eta \in \mathcal{C}^\sharp$ such that*

$$\mu = \xi + \eta . \tag{1}$$

Proof. Define $\Gamma_\mu = \{\gamma \in \mathcal{C} : \gamma \leq_n \mu\}$. Since the zero function belongs to $\mathcal{C}$, Γ_μ is non-empty. Let Γ_o be a totally ordered subset of Γ_μ with respect to the natural ordering $\leq_n$ on V^L and define

$$\gamma_o(c) = \bigvee \{\gamma(c)\colon \gamma \in \Gamma_o\}, \ c \in L .$$

Since $\gamma(c) \leq \gamma(1) \leq \mu(1)$, and V is Dedekind complete, $\gamma_o(c)$ is defined correctly in V. Since Γ_o is totally ordered, the set $D(c) := \{\gamma_o(c) - \gamma(c)\colon \gamma \in \Gamma_o\}$

is directed downwards. Moreover, it is easy to verify that

$$\begin{aligned} 0 &= \gamma_o(c) - \gamma_o(c) = \gamma_o(c) - \bigvee\{\gamma(c)\colon \gamma \in \Gamma_o\} \\ &= -\Big(\bigvee\{-\gamma_o(c) + \gamma(c)\colon \gamma \in \Gamma_o\}\Big) = \bigwedge D(c), \ c \in L\,. \end{aligned}$$

Because the norm of V is order continuous, it follows that the filter $\mathcal{F}(D(c))$ of sections of $D(c)$ converges in norm to 0 for all $c \in L$.

Write $x_\gamma = \gamma_o(1) - \gamma(1)$ for any $\gamma \in \Gamma_o$. Then $\{x_\gamma\colon \gamma \in \Gamma_o\}$ is a net in $D(1)$ such that for every $\gamma_1 \in \Gamma_o$, $\{x_\gamma\colon \gamma \in \Gamma_o, \gamma_1 \leq_n \gamma\} \subseteq D_1$, where D_1 is the section of $D(1)$ determined by x_{γ_1}. Given $\varepsilon > 0$, by [2, §7, no. 1, prop.1], there exists $\gamma_1 \in \Gamma_o$ such that $\|x_\gamma\| < \varepsilon$ whenever $\gamma \in \Gamma_o$ and $\gamma_1 \leq_n \gamma$. This entails that

$$\lim_{\gamma \in \Gamma_o} \big(\gamma_o(1) - \gamma(1)\big) = 0$$

in the norm topology of V. Let now $c \in L$ be arbitrary. Due to inequalities

$$\begin{aligned} 0 &\leq \gamma_o(c) - \gamma(c) = \gamma_o(1) - \gamma(1) - \big(\gamma_o(c^\perp) - \gamma(c^\perp)\big) \\ &\leq \gamma_o(1) - \gamma(1), \end{aligned}$$

we conclude that $\|\gamma_o(c) - \gamma(c)\| \leq \|\gamma_o(1) - \gamma(1)\|$. This implies

$$\lim_{\gamma \in \Gamma_o} \big(\gamma_o(c) - \gamma(c)\big) = 0 \tag{2}$$

uniformly for $c \in L$ in the norm topology of V.

Clearly that $\gamma_o \in V^L$ and $0 \leq_n \gamma_o \leq_n \mu$. We now show that γ_o is finitely additive. Let $a, b \in L$ be mutually orthogonal. Since $\|\gamma_o(a \vee b) - \gamma_o(a) - \gamma_o(b)\| \leq \|\gamma_o(a \vee b) - \gamma(a \vee b)\| + \|\gamma_o(a) - \gamma(a)\| + \|\gamma_o(b) - \gamma(b)\|$ for any $\gamma \in \Gamma_o$, it follows from (2) that $\gamma_o \in a(L, V)_+$, and the uniform closedness of $\mathcal{C}$ gives γ_o is an element of $\mathcal{C}$. This together with $\gamma_o \leq_n \mu$ means that γ_o is a majorant of Γ_o in Γ_μ. It follows from Zorn's lemma that Γ_μ contains a maximal element ξ which belongs to $\mathcal{C}$ and $\xi \leq_n \mu$.

Put $\eta = \mu - \xi$, clearly that $\eta \in a(L, V)_+$. To finish the proof, we show that $\eta \in \mathcal{C}^\sharp$. Let $\gamma \in \mathcal{C}$ be such that $\gamma \leq_n \eta = \mu - \xi$, so that $\gamma + \xi \leq_n \mu$. Because $\gamma + \xi \in \mathcal{C}$, the maximality of ξ in Γ_μ implies $\gamma = 0$. □

We recall that the problem of the uniqueness of decomposition in (1) seems to be open. For a partial result see, e.g., Theorem 7.1.

5. Applications of the decomposition theorem

In the present section, we apply the Decomposition theorem 4.1 to obtain Yosida–Hewitt-type decompositions for special cones of σ-additive measures, completely additive measures, $\mathcal{P}$-regular measures, subadditive measures, etc., as well as Lebesgue-type decompositions.

COROLLARY 5.1. *Every positive finitely additive measure $\mu : L \to V$ can be expressed as a sum $\mu = \xi + \eta$, where ξ is a positive completely additive measure from V^L, and η is a finitely additive measure such that if $\zeta \leq_n \eta$, $\zeta \in ca(L,V)_+$, then $\zeta = 0$.*

Proof. Define the quasi cone $\mathcal{C}_1 = ca(L,V)_+$. We show that $\mathcal{C}_1$ is uniformly closed. So let $\{\mu_t\}$ be a net from $\mathcal{C}_1$ and let $\|\mu_t(c) - \mu(c)\| \to 0$ uniformly for $c \in L$ in the norm topology of V. It is clear that $\mu \in a(L,V)_+$. Suppose that the join $a = \bigvee_{i\in I} a_i$ of mutually orthogonal elements from L exists in L. Let $\varepsilon > 0$ be given. Then there exists μ_{t_o} such that $\|\mu_{t_o}(c) - \mu(c)\| < \varepsilon/3$ for any $c \in L$. Since $\mu_{t_o} \in ca(L,V)_+$, there exists a finite subset J_o of I such that $\|\mu_{t_o}(a) - \sum_{i\in J} \mu_{t_o}(a_i)\| < \varepsilon/3$ whenever J is a finite subset of I containing J_o. Then $\|\mu(a) - \sum_{i\in J}\mu(a_i)\| \leq \|\mu(a) - \mu_{t_o}(a)\| + \|\mu_{t_o}(a) - \sum_{i\in J}\mu_{t_o}(a_i)\| + \|\mu_{t_o}(\bigvee_{i\in J} a_i) - \mu(\bigvee_{i\in J} a_i)\| < \varepsilon$, so that $\mu \in \mathcal{C}_1$.

To the rest it suffices to apply Theorem 4.1. □

COROLLARY 5.2. *Every element $\mu \in a(L,V)_+$ can be expressed as a sum $\mu = \xi + \eta$, where ξ belongs to $\sigma a(L,V)_+$ and $\eta \in \sigma a(L,V)_+^\sharp$.*

Proof. It is identical to the proof of Corollary 5.1 if we use the cone $\mathcal{C}_2 = \sigma a(L,V)_+$. □

REMARK 5.3. Corollaries 5.1 and 5.2 have been proved in [5]. They are analogues of the classical Yosida-Hewitt decomposition. In [5], η from the decomposition corollaries 5.1 and 5.2 are said to be a weakly purely additive measure and a purely additive measure.

Let $\mathcal{P}$ be a non-empty subset of L. We say that a finitely additive measure $\mu \in a(L,V)_+$ is *$\mathcal{P}$-regular* if for any $a \in L$ and any $\varepsilon > 0$ there exists $b \in \mathcal{P}$, $b \leq a$, such that $\|\mu(a) - \mu(b)\| < \varepsilon$. We denote by $a_{\mathcal{P}}(L,V)_+$ the set of all $\mathcal{P}$-regular elements from $a(L,V)_+$.

COROLLARY 5.4. *Let $\mathcal{P}$ be a non-empty set of L such that if $a, b \in \mathcal{P}$, then $a \vee b$ exists in L and belongs to $\mathcal{P}$. Then every element $\mu \in a(L,V)_+$ can be expressed as a sum $\mu = \xi + \eta$, where ξ is a $\mathcal{P}$-regular positive finitely additive measure and $\eta \in a_{\mathcal{P}}(L,V)_+^\sharp$.*

P r o o f . Define the set $\mathcal{C}_3 = a_{\mathcal{P}}(L,V)_+$. Then if $\mu_1,\mu_2 \in \mathcal{C}_3$, then $\mu_1+\mu_2 \in \mathcal{C}_3$. Really, let $a \in L$ and $\varepsilon > 0$ be given. We find $b_1, b_2 \leq a$ such that $\|\mu_i(a) - \mu_i(b_i)\| < \varepsilon/2$. Then $b = b_1 \vee b_2 \in \mathcal{P}$ and $\|\mu_1(a) + \mu_2(a) - \mu_1(b) - \mu_2(b)\| \leq \|\mu_1(a \wedge b^\perp)\| + \|\mu_2(a \wedge b^\perp)\| \leq \|\mu_1(a \wedge b_1^\perp)\| + \|\mu_2(a \wedge b_2^\perp)\| < \varepsilon$.

The uniform closedness of $\mathcal{C}_3$ is now simple. Applying Theorem 4.1, we obtain the decomposition in question. □

Now we introduce the following notions. An element $\mu \in a(L,V)_+$ is said to be (1) a *valuation* if $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \wedge y = 0$ and $x \vee y$ exists in L; (2) *subadditive* if $\mu(x \vee y) \leq \mu(x) + \mu(y)$ whenever $x \vee y \in L$.

We denote by $v(L,V)_+$, $sa(L,V)_+$ the sets of all V-valued valuations and subadditive positive measures, respectively, on L.

COROLLARY 5.5. *Theorem* 4.1 *holds if* $\mathcal{C} = v(L,V)_+$, *or* $\mathcal{C} = sa(L,V)_+$.

P r o o f . It is necessary to verify the conditions of Theorem 4.1. □

We recall that in all above corollaries $\mu \in \mathcal{C}$ iff $\eta = 0$, and $\mu \in \mathcal{C}^\sharp$ iff $\xi = 0$, where ξ, η are from the decomposition (1).

6. Lebesgue–type–decompositions

Let $\mu \in a(L,V)_+$ and let $(W, \|\cdot\|)$ be a normed Riesz space and let $\lambda \in a(L,W)_+$. We say that (i) μ is λ-*continuous*, and we write $\mu \ll_\varepsilon \lambda$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that every $a \in L$ with $\|\lambda(a)\| < \delta$ implies $\|\mu(a)\| < \varepsilon$; (ii) μ is *dominated* by λ, and we write $\mu \ll \lambda$, if $\lambda(a) = 0$ implies $\mu(a) = 0$. It is clear that if $\mu \ll_\varepsilon \lambda$, then $\mu \ll \lambda$. The converse statement holds, for example, if L is a σ-algebra of subsets and μ and λ are real-valued σ-additive measures.

It is known that for orthomodular posets it does not hold, in general. For example, let $L = L(\mathbb{R}^2)$, i.e., the system of all closed subspaces of $\mathbb{R}^2$, and let $V = \mathbb{R} = W$ and let $\{M_n\}$ be a sequence of one-dimensional subspaces of $\mathbb{R}^2$ such that for all $n \neq m$, $M_n \not\perp M_m$. We define $\mu(M_n) = 1/(n+1)$, $\mu(M_n^\perp) = n/(n+1)$, $\mu(\mathbb{R}^2) = 1$, $\mu(0) = 0$, and for all other one-dimensional subspaces M let $\mu(M) = 1/2$, and define $\lambda(M) = 1/2$ if $\dim M = 1$ and $\lambda(\mathbb{R}^2) = 1$, $\lambda(0) = 0$. Then $\mu \ll \lambda$ but $\mu \not\ll_\varepsilon \lambda$.

We say that two measures $\mu \in a(L,V)_+$ and $\lambda \in a(L,W)_+$ are *singular*, and we write $\mu \perp \lambda$, if there exists $a \in L$ such that $\mu(a^\perp) = 0$, and $\lambda(a) = 0$.

We say that μ is λ-*singular* if, whenever $\gamma \in a(L,V)_+$, $\gamma \ll_\varepsilon \lambda$ and $\gamma \leq_n \lambda$, then $\gamma = 0$. We recall that according to [5], if μ is λ-singular, then μ and λ

are singular, and if $V = W$, then $\mu \wedge \nu = 0$, where the meet $\wedge$ is taken in $a(L,V)_+$.

Now we present two Lebesgue-type decompositions.

THEOREM 6.1. ([5]) *Let $\mu \in a(L,V)_+$, $(W, \|\cdot\|)$ be a normed Riesz space and let $\lambda \in a(L,W)_+$. Then μ can be expressed in the form* (1), *where $\xi, \eta \in a(L,V)_+$, $\xi \ll_\varepsilon \lambda$, and η is λ-singular.*

P r o o f. Let us define $\mathcal{C} = \{\gamma \in a(L,V)_+ : \gamma \ll_\varepsilon \lambda\}$. Then $\mathcal{C}$ is a quasi cone of $a(L,V)_+$ which is uniformly closed. Indeed, let $\{\gamma_i\}$ be a net of $\mathcal{C}$ which converges uniformly for $c \in L$ in the norm topology of V. Given $\varepsilon > 0$ we find γ_i such that $\|\gamma(c) - \gamma_i(c)\| < \varepsilon/2$ for all $c \in L$. Since $\gamma_i \ll_\varepsilon \lambda$, we find $\delta > 0$ such that $\|\lambda(a)\| < \delta$ implies $\|\gamma_i(a)\| < \varepsilon/2$. Therefore, for a with $\|\lambda(a)\| < \delta$ we have $\|\gamma(a)\| \le \|\gamma(a) - \gamma_i(a)\| + \|\gamma_i(a)\| < \varepsilon$.

Applying Theorem 4.1, we obtain the decomposition in question. □

Now we present the following generalization of a weak Lebesgue decomposition from [18]:

THEOREM 6.2. *For any pair of finitely additive measures $\mu, \lambda \in a(L,V)_+$ there exist two elements ξ and η in $a(L,V)_+$ such that*

$$\mu = \xi + \eta, \quad \xi \ll \lambda, \tag{3}$$

and $\eta \wedge \lambda = 0$.

P r o o f. The set $\mathcal{C}_\lambda = \{\xi \in a(L,V)_+ : \xi \ll \lambda\}$ is a uniformly closed quasi cone in $a(L,V)_+$. Applying Theorem 4.1, we obtain the first part of (3), where $\eta \in \mathcal{C}_\lambda^\sharp$. Suppose now that κ is an element of $a(L,V)_+$ such that $\kappa \le_n \eta$ and $\kappa \le_n \lambda$. Then $\kappa \in \mathcal{C}_\lambda$ and from the basic property of the set $\mathcal{C}_\lambda^\sharp$ we conclude that $\kappa = 0$, i.e., $0 = \eta \wedge \lambda$. □

We recall that the above Lebesgue-type decompositions can varied if we define different types of quasi cones, for example, quasi cones of σ-additive measures, completely additive measures, $\mathcal{P}$-regular measures, etc.

7. Concluding remarks

It is worth to say that in some particular cases, $\mathcal{C}$ can consist only of the zero function, and in this case, the decomposition (1) is trivial, since $\mathcal{C}^\sharp = a(L,V)_+$. For example, there are cases of L having no system of non-trivial σ-additive or completely additive measures. This case can happen, e.g., if L is a Boolean

σ-algebra and $V = \mathbb{R}$ [19], or, if $L = E(S)$ and $V = \mathbb{R}$ for an incomplete inner product space S because in this case the set of all completely additive measures is $\{0\}$. The former follows from the assertion [10, 11] saying that S is complete iff $E(S)$ possesses at least one non-zero completely additive measure. On the other hand, even for incomplete S, the set of all $\mathcal{P}(S)$-regular finitely additive measures from $a\big(E(S),\mathbb{R}\big)_+$ is not trivial, where $\mathcal{P}(S)$ is the set of all finite-dimensional subspaces of S. Indeed, according to [9, 10], the following Aarnes decomposition holds (see also [14, 15] in other structures):

THEOREM 7.1. *For any real-valued finitely additive measure μ on $E(S)$ there exists a unique decomposition*

$$\mu = \xi + \eta,$$

where ξ is a $\mathcal{P}(S)$-regular finitely additive measure, and η is a real-valued finitely additive measure vanishing on all finite-dimensional subspaces of S.

μ is regular if and only, if there is a Hermitian trace operator $T\colon \bar{S} \to \bar{S}$, where $\bar{S}$ denotes the completion of S, such that

$$\mu(M) = \operatorname{tr}(TP_{\overline{M}}), \quad M \in E(S), \tag{4}$$

where $P_{\overline{M}}$ denotes the orthoprojector from $\bar{S}$ onto $\overline{M}$.

Proof. Let $\mathcal{C}$ be the quasi cone of all $\mathcal{P}(S)$-regular measures on $E(S)$. Then, according to [9, 10], $\mathcal{C}^\sharp$ is the set of all elements from $a(E(S),\mathbb{R})_+$ which vanishes on every finite-dimensional subspace of S. □

It is worth to say that (4) is the Gleason formula for the set of all splitting subspaces. We recall that if S is complete, then μ is $\mathcal{P}(S)$-regular iff μ is completely additive, and this is equivalent to the so-called Gleason's formula (4). The decomposition from Theorem 7.1 gives an important one for $E(S)$ even when $ca(E(S),\mathbb{R}) = \{0\}$ for incomplete S.

Finally, let $\mathcal{A}$ be a von Neumann algebra of operators acting on a Hilbert space H. Denote by $L_{\mathcal{A}}(H)$ the set of all orthoprojectors from $\mathcal{A}$. Then $L = L_{\mathcal{A}}(H)$ is a complete orthomodular lattice, where the partial ordering $\le$ is defined via $P \le Q$ iff $(Px,x) \le (Qx,x)$, $x \in H$, and the orthocomplementation $\perp$ is $P^\perp := I - P$, where I is the identity operator on H.

Suppose that V is a Banach lattice with an order continuous norm $\|\cdot\|$. Using the quasi cone $ca(L_{\mathcal{A}}(H),V)_+$, we can any element $\mu \in a(L_{\mathcal{A}}(H),V)_+$ decompose in the form $\mu = \xi + \eta$, where $\xi \in ca(L_{\mathcal{A}}(H),V)_+$ and $\eta \in ca(L_{\mathcal{A}}(H),V)_+^\sharp$, Corollary 5.1. According to [3] (see also the sketch of the proof for complex-valued measures in [16]), μ, ξ and η can be extended to unique bounded linear operators $\hat{\mu}, \hat{\xi}$ and $\hat{\eta}$ from $\mathcal{A}$ into V such that $\mu = \hat{\mu}|L_{\mathcal{A}}(H)$, $\xi = \hat{\xi}|L_{\mathcal{A}}(H)$, and $\eta = \hat{\eta}|L_{\mathcal{A}}(H)$. In this case we have $\hat{\mu} = \hat{\xi} + \hat{\eta}$.

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