Tatra Mountains Math. Publ. **3** (1993), 213–230

TATRA MOUNTAINS
Mathematical Publications

RIESZ SPACES, INTEGRATION AND SANDWICH THEOREMS

ANTONIO BOCCUTO

ABSTRACT. Some definitions of integrals are introduced, for functions taking values in suitable Riesz spaces and with respect to finitely additive measures, in such a way that every bounded function is integrable. Furthermore, some Vitali-type convergence theorems are proved. Finally, some sandwich-type and Hahn-Banach type theorems are given, for finitely additive maps with values in $R \cup \{\infty\}$, where R is a Dedekind complete Riesz space and ∞ is an "extra" element, by convention greater than all elements of R.

1. Introduction

In the literature, there are many studies about integration for mappings, taking values in a Riesz space R. Among the authors, we recall McGill [11], Orihara and Sunouchi [13], Riečan [15]. In this paper, we construct some types of integral for functions defined on an arbitrary set G, and taking values in a suitable Riesz space R, with respect to finitely additive means. By using these constructions, we obtain that every bounded function is integrable (Section 2). In Section 3, we prove some convergence theorems for this kind of integrals. In Section 4, by using the introduced integrals, we give a sandwich-type theorem for invariant measures, which is a generalization of the results obtained in [1] and [2]. Furthermore, a characterization of amenable semigroups is given.

2. Constructions of integrals

In this section, we construct some types of integrals for maps, defined on an arbitrary set G and taking values in a suitable Riesz space R, with respect to a mean $\mu\colon \mathcal{P}(G) \to [0,1]$ ("mean" for us is any finitely additive probability).

The first integral is inspired by a technique due to Chojnacki [3].

AMS Subject Classification (1991): 28A70, 43A07.
Key words: Riesz spaces, π-spaces, integration, sandwich theorems.

Let R be any Dedekind complete Riesz space. By Maeda–Ogasawara–Vulikh representation theorem (see [5]), there exists a compact and Stonian (i.e., extremally disconnected) topological space Ω, unique up to homeomorphisms, such that R can be embedded as a solid subspace of $\mathcal{C}_\infty(\Omega) = \{f : \Omega \to \mathbb{R}$: f is continuous, and $\{\omega \in \Omega: |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. First of all, we consider the case $R = \mathcal{C}(\Omega) = \{f \in \mathbb{R}^\Omega,\ f$ continuous$\}$, where Ω is compact and Stonian.

By virtue of a classical result (see [8], [14]), there exists a discrete set D, such that Ω is homeomorphic to a retract of βD: for example, we can choose D as Ω itself, but endowed with the discrete topology. We denote by r the retraction and identify Ω with this retract.

Now, given a bounded map $f \in \mathcal{C}(\Omega)^G$, let $f_r : G \to \mathcal{C}(\beta D)$ be the map, defined by setting: $f_r(g)(\xi) = f(g)\big(r(\xi)\big)$, $\forall \xi \in \beta D$, $\forall g \in G$. By a classical result, for each $g \in G$, the map $\xi \mapsto f(g)\left(r(\xi)\right)$ is the unique continuous extension to βD of the function $d \mapsto f(g)\left(r(d)\right)$, defined on D. Now, for every $d \in D$, let $I_f^*(d) = \int_G f(g)\left(r(d)\right) d\mu(g)$. The map $d \mapsto I_f^*(d)$ has a unique continuous extension $\xi \mapsto \hat{I}_f(\xi)$ to βD. Our integral, $I_f = \int_G f(g)\, d\mu(g)$, will be the restriction of $\hat{I}_f$ to Ω. It is easy to check that I_f is a linear positive R-valued functional, and that

$$\int_{A\cup B} f(g)\, d\mu(g) = \int_A f(g)\, d\mu(g) + \int_B f(g)\, d\mu(g), \tag{2.1}$$

for each bounded function $f \in R^G$, and for each pair of disjoint sets $A, B \subset G$. Moreover, it is easy to see that:

$$\text{If} \quad f(g) = r \quad \forall g \in G, \quad \text{then} \quad I_f = r \tag{2.2}$$

and that, if G is an amenable semigroup and $\mu: \mathcal{P}(G) \to [0,1]$ is right[left]-invariant, and $f \in R^G$ is bounded,

$$I_f = I_{f_h}\,[I_f = I_{f_{0,h}}], \text{ where } f_h\,[f_{0,h}]: G \to R \text{ is defined by setting :}$$
$$f_h(g) = f(gh)\left[f_{0,h}(g) = f(hg)\right]. \tag{2.3}$$

Now, let $R = \mathcal{C}_\infty(\Omega)$, where Ω is compact and Stonian. For each positive bounded $f \in \mathcal{C}_\infty(\Omega)^G$, put $\overline{I}_f = \sup_{m\in\mathbb{N}} I_{f\wedge \underline{m}}$, where $\underline{m}$ is the element of $\mathcal{C}(\Omega)$, which associates the constant $m \in \mathbb{N}$ with every element $\omega \in \Omega$, and $f \wedge \underline{m}: G \to \mathcal{C}(\Omega)$ is defined by setting: $(f \wedge \underline{m})(g) = \left[f(g)\right] \wedge \underline{m}$, $\forall g \in G$. It is easy to prove that $f \mapsto \overline{I}_f$ is a monotone, additive and positively homogeneous map, and hence it can be uniquely extended and defined linearly on the whole set $\{f \in \mathcal{C}_\infty(\Omega)^G : f$ is bounded$\}$. We will denote by I_f this extension; $I_f =$

$\int_G f(g)\,d\mu(g)$ will be our integral (C-integral) of f. One can see easily that I_f is a linear monotone R-valued functional, satisfying (2.1), (2.2) and (2.3).

In the general case, that is when R is embedded as a solid subspace of $\mathcal{C}_\infty(\Omega)$, this construction still makes sense: indeed, I_f is an element of R, by solidity of R.

Now, we introduce another type of integral. Given any Riesz space R, let $R^* = \{f \in \mathbb{R}^R : f \text{ is a linear functional}\}$, $R^+ = \{f \in R^* : f \text{ is order bounded}\}$, $R^\times = \{f \in R^* : f \text{ is order-continuous}\}$, and let $c\colon R \to R^{+*}[R^{\times *}]$ be the evaluation map, defined by setting $c(r)(y) = y(r)$, $\forall r \in R$, $\forall y \in R^+[R^\times]$. The following property holds (see [6]).

PROPOSITION 2.4. *The evaluation map c is a Riesz homomorphism from R into $R^{+\times}[R^{\times\times}]$.*

From now on, we put $Y = R^\times$ (analogous results hold, if we put $Y = R^+$). Let G be any set, and assume that R is any Riesz space. For every bounded map $f \in R^G$, let $\hat{f}\colon G \to Y^\times$ defined by setting: $\hat{f}(g)(y) = c(f(g))(y)$, $\forall g \in G$ and $\forall y \in Y$.

It is easy to see that, for all $y \in Y$, the map $f_y \in \mathbb{R}^G$ defined by putting $f_y(g) = \hat{f}(g)(y)$ is bounded, and hence we can define $\hat{I}_f(y) = \int_G f_y(g)\,d\mu(g)$, for each bounded map $f \in R^G$, $\forall y \in Y$. $\hat{I}_f$ is a linear functional; moreover, as $Y^\times$ is solid in Y^+ (see also [6]) and Y^+ is solid in Y^*, $\hat{I}_f$ is $Y^\times$-valued. We will call $\hat{I}_f$ the *Dunford integral* (D-*integral*) of f.

DEFINITION 2.5. A Riesz space R is said to be a π*-space* [$+$*-space*] if the evaluation map $c\colon R \to R^{\times\times}$ $[R^{+\times}]$ is one-to-one.

DEFINITION 2.6. Let G be any set, and let R be a π-space [$+$-space]. A bounded mapping $f \in R^G$ is said to be *Pettis-integrable* (P-*integrable*) if its Dunford integral $\hat{I}_f$ is an element of $c(R)$. If this is the case, $I_f = c^{-1}(\hat{I}_f)$ will be called the *Pettis integral* (P-*integral*).

We note that, if R is a Dedekind complete π-space, every bounded function $f \in R^G$ is P-integrable: in fact, as $c(R)$ is solid in $R^{\times\times}$ (see also [6]), then $\hat{I}_f \in c(R)$. The P-integral for Dedekind complete π-spaces will be called the π*-integral.* It is easy to see that (2.1), (2.2) and (2.3) hold.

Now, we construct another type of integral. Let R be an Archimedean space.

DEFINITION 2.7. A subset K of a Riesz space R is said to be *order dense* in R if, for each $r \in R$, such that $r \geq 0$ and $r \neq 0$, there exists $k \in K$ such that $0 \leq k \leq r$ and $k \neq 0$.

Let $\mathcal{F}$ be the class of all order dense ideals of R, and set $\Phi = \bigcup_{I \in \mathcal{F}} I^\times$. Then, a function $\varphi \in \mathbb{R}^R$ belongs to Φ if there exists an order dense ideal I of R such that φ is an order-continuous linear functional on I.

On Φ, we define the following relation: $\varphi_1 \sim \varphi_2$ when the set $\{r \in R\colon \varphi_1(r) = \varphi_2(r)\}$ contains an order dense ideal of R (see also [9]). As the intersection of a finite number of order dense ideals is still an order dense ideal (see [10]), then $\sim$ is an equivalence relation.

Now, put $R^\rho \equiv \Phi/\sim$. We define a sum, a product and an order relation on R^ρ in the following way: $[\varphi_1] + [\varphi_2] = [\varphi_3]$ when there exist $\varphi\prime_1 \in \varphi_1$, $\varphi\prime_2 \in \varphi_2$, $\varphi\prime_3 \in \varphi_3$ such that the set $\{r\colon \varphi\prime_1(r) + \varphi\prime_2(r) = \varphi\prime_3(r)\}$ contains an order dense ideal (it is easy to check that this definition does not depend on $\varphi\prime_1$, $\varphi\prime_2$, $\varphi\prime_3$, and that $[\varphi_1 + \varphi_2] = [\varphi_1] + [\varphi_2]$). Analogously we define $\alpha \cdot [\varphi]$, for every $[\varphi] \in R^\rho$ and for all $\alpha \in \mathbb{R}$. Furthermore, we shall say that $[\varphi] \geq 0$ if there exists $\varphi\prime \in [\varphi]$ such that the set $\{r \in R\colon \varphi(r) \geq 0\}$ contains an order dense ideal of R, and $[\varphi_1] \leq [\varphi_2]$ if $[\varphi_2] - [\varphi_1] \geq 0$.
Thus, R^ρ is a Dedekind complete Riesz space (see [9]).

Given an Archimedean Riesz space R, we define the evaluation map $c\colon R \to R^{\rho\rho}$ in the following way (see [9]). For every $r \in R$, $r \geq 0$, set $I_r = \{[\varphi] \in R^\rho\colon r \in D_\varphi, \varphi \in [\varphi]\}$, where, for each φ such that $[\varphi] \in R^\rho$, D_φ is the greatest order dense ideal of R on which $|\varphi|$ can be extended as a real-valued map.

$$\begin{aligned} &\text{Let}\quad c(r)([\varphi]) = \varphi(r),\ \forall r \geq 0 \quad \text{and for all} \quad \varphi \quad \text{such that} \\ &[\varphi] \in I_r, \quad \text{and put} \quad c(r) = c(r^+) - c(r^-),\ \forall r \in R. \end{aligned} \tag{2.8}$$

It is easy to prove that I_r is an order dense ideal of R^ρ, that c is well-defined and c is a Riesz homomorphism (see also [9]).

Definition 2.9. An Archimedean Riesz space R is called a *ρ-space* if the evaluation map c defined in (2.8) is one-to-one.

Now, let R be a ρ-space, and assume that $f \in R^G$ is a bounded function. Then, there exist $r_1, r_2 \in R$ such that $0 \leq [f(g)]^+ \leq r_1$, $0 \leq [f(g)]^- \leq r_2$, $\forall g \in G$. By identifying r_1 with $c(r_1)$ and r_2 with $c(r_2)$, there exists an order dense ideal I of R^ρ such that $r_1|_I$, $r_2|_I$ are two linear order-continuous real-valued functionals. Then, $r_1|_I$ and $r_2|_I$ are monotonic on the whole of I. Now, fix $g \in G$. Then, there exists an order-dense ideal I_g of R^ρ such that $0 \leq [f(g)]^+(y) \leq r_1(y)$, $\forall y \in I_g$, $y \geq 0$. Without loss of generality, we can suppose $I_g \subset I$. Let $\overline{r_1}^{(g)}(y) \equiv \overline{r_1}(y) = \sup\{r_1(u)\colon 0 \leq u \leq y$, $u \in I_g\}$, $\overline{[f(g)]^+}(y) = \sup\{f(g)(u)\colon 0 \leq u \leq y,\ u \in I_g\}$, $\forall y \in R^\rho$, $y \geq 0$. One can check that $\overline{r_1}(y_1 + y_2) = \overline{r_1}(y_1) + \overline{r_1}(y_2)$, $\alpha\overline{r_1}(y_1) = \overline{r_1}\,\alpha(y_1)$, and $[y_1 \leq y_2] \Rightarrow [\overline{r_1}(y_1) \leq \overline{r_1}(y_2)]$, $\forall y_1, y_2 \in R^\rho$, y_1, $y_2 \geq 0$, and $\forall \alpha \in R_0^+$); similar results hold for $\overline{[f(g)]^+}$. Moreover, we have: $0 \leq r_1(u) \leq r_1(y)$, for all $y \in I$, $y \geq 0$, and for every $u \in I_g$, $0 \leq u \leq y$; taking the supremum, we obtain: $0 \leq \overline{r_1}(y) \leq r_1(y)$. As $0 \leq [f(g)]^+(u) \leq r_1(u)$, $\forall u \in I_g$, $u \geq 0$, then $0 \leq \overline{[f(g)]^+}(y) \leq \overline{r_1}(y) \leq r_1(y)$, for each $y \in R^\rho$, $y \geq 0$. Thus, $\overline{[f(g)]^+}$ is an additive, monotone, positively homogeneous real-valued functional on the set of all positive elements of $\overline{I\text{, and}}$ then it can be extended linearly on I: we indicate this extension by $\overline{[f(g)]^+}$ again. As r_1 is order-continuous on I, then $\overline{[f(g)]^+}$ is too. Analogously, we can construct $\overline{[f(g)]^-}$, which will be an order-continuous linear real-valued functional on I.

Now, $f(g)$ is equivalent to $\overline{[f(g)]^+} - \overline{[f(g)]^-}$; by identifying these two objects, we can state that $f(g)$ is an order-continuous linear real-valued functional on I, $\forall g \in G$, and $-r_2(y) \leq f(g)(y) \leq r_1(y)$, $\forall y \in I$, $y \geq 0$, and $\forall g \in G$. So, it is possible to define $J_f(y) = \int_G f(g)(y)\, d\mu(g)$, $y \in I$, for all bounded functions $f \in R^G$. It is easy to check that J_f is a linear real-valued functional on I. As $-r_2$ and r_1 are order-continuous on I, so is J_f. By identifying J_f with its class of equivalence, J_f will be an element of $R^{\rho\rho}$. It is easy to prove that $f \mapsto J_f$ is a linear monotone $R^{\rho\rho}$-valued functional. If R is Dedekind complete, then $c(R)$ is a solid subspace of $R^{\rho\rho}$ (see [9]), and thus $J_f \in c(R)$, for each bounded function $f \in R^G$.

Now, let R be a Dedekind complete ρ-space, and set $I_f = c^{-1}(J_f)$, for every bounded function $f \in R^G$. We will call I_f the *ρ-integral* of f. One can see easily that I_f is well-defined, and the map $f \mapsto I_f$ is a linear monotone R-valued functional, satisfying (2.1), (2.2) and (2.3). We note that every π-space is a ρ-space, but there exist Dedekind complete ρ-spaces which are not π-spaces (see [4], [5]).

3. Integrability and convergence theorems

In this section, we introduce a concept of integrability for functions which are not necessarily bounded, and in such a way that every bounded map is integrable; moreover, we prove some convergence-type theorems.

From now on, our integral will be the C-integral, the π-integral or the ρ-integral (and we will not write it explicitly).

Let R be a Dedekind complete Riesz space, $R \neq \{0\}$.

Definition 3.1. A *unit* of R is an element $u \in R$ such that $u \geq 0$, $u \neq 0$.

Definition 3.2. Let u be a unit of R. We will say that the sequence $\{r_n\}_n$ *u-converges* to $r \in R$ if, $\forall \varepsilon \geq 0$, $\exists \overline{n}(\varepsilon) \in \mathbb{N}$ such that $|r_n - r| \leq \varepsilon u$, $\forall n \geq \overline{n}$. We say that $\{r_n\}_n$ *converges relatively* to r if it u-converges for some unit u (see also [10]).

Definition 3.3. A sequence of functions $\{f_n\}_n \in R^G$ *u-converges uniformly* to $f \in R^G$ if, $\forall \varepsilon \geq 0$, $\exists \overline{n}(\varepsilon) \in \mathbb{N}$ such that $|f_n(g) - f(g)| \leq \varepsilon u$, $\forall n \geq \overline{n}$ and $\forall g \in G$.

Definition 3.4. A sequence $\{f_n\}$, $f_n \in R^G$ *converges in measure* to $f \in R^G$ if, for some unit $u \in R$, it holds: $\forall \varepsilon$, $\sigma > 0$, $\exists \overline{n}(\varepsilon, \sigma)$ such that, $\forall n \geq \overline{n}$,

$\mu(A_n^\varepsilon) < \sigma$, where $A_n^\varepsilon = \{g \in G \colon |f_n(g) - f(g)| \not\leq \varepsilon u\}$.

Obviously, every u-uniformly convergent sequence is convergent in measure. The following results hold (the proof is easy).

PROPOSITION 3.5. *If $\{f_n\}_n, \{h_n\}_n \in R^G$ are two sequences of functions, convergent in measure to f, h respectively, then $\{f_n + h_n\}_n$ converges in measure to $f + h$.*

PROPOSITION 3.6. *If $\{f_n\}_n$ converges in measure to f, and $\alpha \in \mathbb{R}$, then $\{\alpha f_n\}_n$ converges in measure to αf.*

PROPOSITION 3.7. *If $\{f_n\}_n$ converges in measure to f, then $\{|f_n|\}_n$ converges in measure to $|f|$.*

Now, let f_n, f be bounded elements of R^G.

DEFINITION 3.8. We say that a sequence $\{f_n\}_n$, $f_n \in R^G$, *converges in L^1* to $f \in R^G$ if, for some unit $u \in R$, one has that, $\forall \varepsilon > 0$, $\exists \overline{n}(\varepsilon) \in \mathbb{N}$ such that:

$$\int_G |f_n(g) - f(g)| \, d\mu(g) \leq \varepsilon u, \tag{3.8.1}$$

$$\mu(A_n^\varepsilon) < \varepsilon, \quad \forall n \geq \overline{n}. \tag{3.8.2}$$

We note that, in general, (3.8.1) does not imply (3.8.2).

EXAMPLE 3.8.3. Let $G = [0,1]$, $\mu \colon \mathcal{P}(G) \to [0,1]$ be a finitely additive extension of Lebesgue measure. Take $R = \mathcal{C}(\beta D)$, where $D = [0,1]$, endowed with the discrete topology. Now, we indicate by $\oplus$ the sum mod 1 of two elements of $[0,1]$, and define

$$f_n(g)(d) = \begin{cases} 1, & \text{if } g \in [d, d \oplus \frac{1}{n}], \\ 0, & \text{otherwise,} \end{cases}$$

$\forall n \in \mathbb{N}$, $\forall g \in G$, $\forall d \in D$.

For all $n \in \mathbb{N}$ and for every $g \in G$, the map $d \mapsto f_n(g)(d)$ is $\{0,1\}$-valued, and then it has a unique continuous $\{0,1\}$-valued extension, defined on the whole of βD. We have: $\int_G f_n(g)(d) \, d\mu(g) = \frac{1}{n}$, $\forall d \in D$, $\forall n \in \mathbb{N}$. Taking the C-integral, we obtain, $\forall n \in \mathbb{N}$: $\int_G f_n(g) \, d\mu(g) = \underline{\frac{1}{n}}$, where $\underline{\frac{1}{n}}$ is the mapping which associates the constant $\frac{1}{n} \in \mathbb{R}$ with every $\xi \in \beta D$. Thus, (3.8.1) holds, with $f \equiv 0$ and $u = \underline{1}$. Every unit u of $\mathcal{C}(\beta D)$ is a bounded real-valued function; thus, for all unit u, there exists $\overline{\varepsilon} = \overline{\varepsilon}(u) \in \,]0,1[$ such that $\overline{\varepsilon} u \leq \underline{\frac{1}{2}}$.

For each $n \in \mathbb{N}$, it is:

$$\mu(\{g \in G \colon f_n(g) \not\leq \overline{\varepsilon} u\}) \geq \mu\Big(\{g \in G \colon f_n(g) \not\leq \underline{\frac{1}{2}}\}\Big) = \mu(G) > \overline{\varepsilon},$$

and hence (3.8.2) does not hold.

DEFINITION 3.9. A sequence $\{f_n\}_n$ is said to be *uniformly integrable* if, for some unit $w \in R$, it holds:

$$\forall \varepsilon > 0,\ \exists \overline{n}(\varepsilon) \in \mathbb{N} \quad \text{and} \quad \exists\, \delta(\varepsilon) > 0$$

such that

$$\int_A |f_n(g)|\, d\mu(g) \leq \varepsilon w, \quad \forall n \geq \overline{n} \quad \text{and} \quad \forall A \subset G \quad \text{with} \quad \mu(A) < \delta\,.$$

Now, we prove the following:

THEOREM 3.10. *Let $f_n, f \in R^G$, be bounded functions. Then the following statements are equivalent:*

(3.10.1) *$\{f_n\}_n$ converges in measure to f and is uniformly integrable.*
(3.10.2) *$\{f_n\}_n$ converges in L^1 to f.*

Proof. (3.10.1) $\Rightarrow$ (3.10.2).
By hypothesis, f is bounded, and hence there exists a unit $v \in R$, such that $|f(g)| \leq v$, $\forall g \in G$. Then, we have

$$\int_A |f(g)|\, d\mu(g) \leq \int_A v\, d\mu(g) = \mu(A) \cdot v \leq \varepsilon v\,,$$

for every subset $A \subset G$, such that $\mu(A) \leq \varepsilon$.

Now, fix $\varepsilon > 0$, and pick $\overline{m}(\varepsilon)$ and $\delta(\varepsilon)$ such that the uniform integrability is satisfied. By convergence in measure, we see that, for ε and $\sigma = \min(\varepsilon,\, \delta(\varepsilon))$, there exists $\overline{n}$ such that $\mu(A_n^\varepsilon) < \sigma$, $\forall n \geq \overline{n}$, and $\overline{n}$ can be chosen greater than $\overline{m}$. Thus, $\forall n \geq \overline{n}$, it is:

$$\int_G |f_n(g) - f(g)|\, d\mu(g) = \int_{(A_n^\varepsilon)^c} |f_n(g) - f(g)|\, d\mu(g) + \int_{A_n^\varepsilon} |f_n(g) - f(g)|\, d\mu(g) \leq$$

$$\leq \varepsilon u + \int_{A_n^\varepsilon} |f_n(g) - f(g)|\, d\mu(g) \leq \varepsilon u + \int_{A_n^\varepsilon} |f_n(g)|\, d\mu(g) +$$

$$+ \int_{A_n^\varepsilon} |f(g)|\, d\mu(g) \leq \varepsilon u + \varepsilon w + \varepsilon v\,.$$

(3.10.2) $\Rightarrow$ (3.10.1).

It is obvious that convergence in L^1 implies convergence in measure. Now, let $\{f_n\}_n$ be convergent in L^1 to f. Then, for some unit $u \in R$, we have that, $\forall \varepsilon > 0$, there exists $\overline{n}(\varepsilon) \in \mathbb{N}$ such that $\int_G |f_n(g) - f(g)|\, d\mu(g) \leq \varepsilon u$,

$\forall n \geq \overline{n}$. Moreover, there exists a unit v such that, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that $\mu(A) < \delta \Rightarrow \int_A |f(g)|\, d\mu(g) \leq \varepsilon v$. Thus, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ and $\exists \overline{n}(\varepsilon)$ such that

$$\int_A |f_n(g)|\, d\mu(g) \leq \int_A |f_n(g) - f(g)|\, d\mu(g) + \int_A |f(g)|\, d\mu(g) \leq$$

$$\leq \int_G |f_n(g) - f(g)|\, d\mu(g) + \int_A |f(g)|\, d\mu(g) \leq \varepsilon(u + v)$$

holds true, for all $n \geq \overline{n}$ and all $A \subset G$, with $\mu(A) < \delta$. □

Now, we introduce a concept of integrability.

Definition 3.11. A map $f \in R^G$ is *integrable* if there exists a sequence of bounded functions $\{f_n\}_n$ such that $\{f_n\}_n$ converges in measure to f, and $\{f_n\}_n$ is uniformly integrable.

Given an integrable map $f \in R^G$, we put

$$\int_A f(g)\, d\mu(g) = \lim_{n \to \infty} \int_A f_n(g)\, d\mu(g), \quad \forall A \subset G,$$

where the limit is intended with respect to the relative convergence. We prove the following:

Theorem 3.12. *Let $f \in R^G$ be an integrable function, and $\{f_n\}_n$ as in Definition 3.11. Then the limit $\lim_{n\to\infty} \int_A f_n(g)\, d\mu(g)$ exists uniformly with respect to $A \subset G$ and does not depend on the choice of $\{f_n\}_n$.*

P r o o f. Let $\{f_n^1\}_n$, $\{f_m^2\}_m$ be two uniformly integrable sequences of bounded maps, convergent in measure to the same limit f. Analogously to Theorem 3.10, we can prove that for some unit $e_1, e_2 \in R$, it holds: $\forall \varepsilon > 0$, $\exists \overline{n}(\varepsilon) \in \mathbb{N}$ such that, $\forall m, n \geq \overline{n}$, $\forall A \subset G$,

$$\left| \int_A f_n^i(g)\, d\mu(g) - \int_A f_m^i(g)\, d\mu(g) \right| \leq$$

$$\leq \int_G |f_n^i(g) - f_m^i(g)|\, d\mu(g) \leq \varepsilon\, e_i \leq \varepsilon\, (e_1 + e_2) \quad (i = 1, 2).$$

As R is Dedekind complete, the sequences $\left\{\int_A f_n^i(g)\, d\mu(g)\right\}_n$, $(i = 1, 2)$ are $(e_1 + e_2)$-convergent (see [10]), uniformly with respect to A. Thus, the limits $\lim_{n\to\infty} \int_A f_n^i(g)\, d\mu(g)$, $i = 1, 2$, exist uniformly with respect to $A \subset G$; we denote these quantities by $l_1(A)$ and $l_2(A)$.

Now, put $p_n(g) = |f_n^1(g) - f_n^2(g)|$, $\forall A \subset G$, $\forall n \in \mathbb{N}$. As $\{f_n^1\}$ and $\{f_n^2\}$ are uniformly integrable, it follows that $\{p_n\}$ is uniformly integrable. Moreover, $\{f_n^1\}$ and $\{f_n^2\}$ converge in measure to f, so $\{p_n\}$ converges in measure to 0. Hence, by Theorem 3.10, $p_n \to 0$ in L^1, that is $\exists$ unit r: $\forall \varepsilon > 0$, $\exists \overline{n}$ satisfying $\int_G p_n(g)\, d\mu(g) \le \varepsilon r$, $\forall n \ge \overline{n}$, and thus $|\int_A f_n^1(g)\, d\mu(g) - \int_A f_n^2(g)\, d\mu(g)| \le \varepsilon r$, $\forall A \subset G$, from which it follows:

$$\sup_A |l_1(A) - l_2(A)| \le \left| \int_A f_n^1(g)\, d\mu(g) - l_1(A)\right| + |l_2(A) - $$

$$- \int_A f_n^2(g)\, d\mu(g)| + \left| \int_A f_n^1(g)\, d\mu(g) - \int_A f_n^2(g)\, d\mu(g) \right|.$$

So, taking n sufficiently large, we get

$$\sup_A |l_1(A) - l_2(A)| \le \varepsilon v + 2\varepsilon\,(e_1 + e_2)\,,$$

which gives $l_1(A) = l_2(A)\, \forall A$, by arbitrariness of $\varepsilon > 0$. □

It is easy to prove the following:

Lemma 3.13. *Let R be a Dedekind complete Riesz space, let $a \in R$, and suppose that $\{r_n\}_n$ is a sequence of elements of R, such that $r_n \le a$, $\forall n \in \mathbb{N}$, and such that $\{r_n\}_n$ relatively converges to r. Then, $r \le a$.*

Now, we prove the following:

Theorem 3.14. *Let $f \in R^G$ an integrable map. Then, for some unit u, it holds: $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that $\int_A |f(g)\, d\mu(g)| \le \varepsilon v$, $\forall A \subset G$ with $\mu(A) < \delta$. (Absolute continuity)*

Proof. If $\{f_n\}_n$ is as in Definition 3.11, we have, for some unit v: $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that $\int_A |f_n(g)|\, d\mu(g) \le \varepsilon v$ for every $n \ge \overline{n}$ and for each A with $\mu(A) < \delta$.

By Lemma 3.13, one has:

$$\int_A |f(g)|\, d\mu(g) = \lim_{n \to \infty} \int_A |f_n(g)|\, d\mu(g) \le \varepsilon v\,,$$

and thus the assertion follows. □

Now, we can formulate the definitions of convergence in L^1 and uniform integrability, by requiring simply that the maps f_n and f are integrable. By using Theorem 3.14, and arguing as in the proof of Theorem 3.10, one can prove the following:

THEOREM 3.15. *Let f_n, $f \in R^G$ integrable functions. Then the following statements are equivalent:*

(3.15.1) *$\{f_n\}_n$ converges in measure to f and is uniformly integrable,*
(3.15.2) *$\{f_n\}_n$ converges in L^1 to f.*

The following theorem is a consequence of Theorem 3.15.

THEOREM 3.16. (Dominated convergence theorem). *Let $\{f_n\}_n$ be a sequence of integrable functions, $f_n \in R^G$, and suppose that there exists an integrable function $h \in R^G$ such that $|f_n(g)| \leq |h(g)|$ for all $n \in \mathbb{N}$ and almost everywhere with respect to g. Assume that $\{f_n\}_n$ converges in measure to f. Then $\{f_n\}$ converges in L^1 to f.*

Remark 3.17. In Theorem 3.16, the hypothesis of integrability of f_n can be dropped: indeed, the functions f_n are integrable, by virtue of the following:

THEOREM 3.18. *Let $f, h \in R^G$, and assume that h is an integrable map, and $|f(g)| \leq |h(g)|$ for almost all $g \in G$. Then, f is integrable.*

Without loss of generality, we may suppose that $|f(g)| \leq |h(g)|$, $\forall g \in G$. The general case will be a consequence of this lemma, whose proof is easy.

LEMMA 3.19. *Let $f \in R^G$ be an integrable function , and assume that $f' \in R^G$ is such that $f(g) = f'(g)$, for almost all $g \in G$. Then f' is integrable, and $\int_A f(g)\, d\mu(g) = \int_A f'(g)\, d\mu(g)$, $\forall A \subset G$.*

Proof of Theorem 3.18. Without any restriction, we can suppose that h and f are positive.

Let $\{h_n\}_n$ be a uniformly integrable sequence of bounded functions, such that $\{h_n\}_n$ converges in measure to h. For every $g \in G$, $n \in \mathbb{N}$, put $k_n(g) = h_n(g) \wedge f(g)$. It is easy to check that $\{k_n\}_n$ is a uniformly integrable sequence of bounded functions.

Now, we prove that $\{k_n\}_n$ converges in measure to f. Let u be any unit for which the convergence in measure of $\{h_n\}_n$ to h holds, and fix any $\varepsilon > 0$. Then:

$$\begin{aligned}&\{g \in G \colon |f(g) - h_n(g) \wedge f(g)| \not\leq \varepsilon u\} = \\ &= \{g \in G \colon |h(g) \wedge f(g) - h_n(g) \wedge f(g)| \not\leq \varepsilon u\} \subset \\ &\subset \{g \in G \colon |h(g) - h_n(g)| \not\leq \varepsilon u\}, \qquad \forall n \in \mathbb{N}:\end{aligned}$$

in fact, if $|h(g) - h_n(g)| \leq \varepsilon u$, then

$$|h(g) \wedge f(g) - h_n(g) \wedge f(g)| \leq |h(g) - h_n(g)| \leq \varepsilon u$$

(see also [10]).

Hence, if n is sufficiently large, we get

$$\mu(\{g \in G \colon |f(g) - h_n(g) \wedge f(g)| \not\leq \varepsilon u\}) \leq \mu(\{g \in G \colon |h(g) - h_n(g)| \not\leq \varepsilon u\}) < \varepsilon,$$

and the proof is completed. □

4. Sandwich theorems

Now, we prove a sandwich-type theorem, which is an application of the C-integral and is a generalization of other sandwich theorems (see [1], [2]). We begin with the following:

DEFINITION 4.1. Given any Riesz space R, add to it an extra element, ∞, extending ordering and operations, in such a way that

$$\left.\begin{array}{l} \infty \geq r, \quad \forall\, r \in R, \\ 0 \cdot \infty = 0, \\ \lambda \cdot \infty = \infty, \quad \forall\, \lambda \in \mathbb{R}^+, \\ \infty + r = \infty, \quad \forall\, r \in R \cup \{\infty\}. \end{array}\right\} \tag{4.1.1}$$

Now, we state the sandwich theorem.

THEOREM 4.2. *Let X be a preordered Abelian monoid, $G \subset X^X$ a right-amenable semigroup of monotone homomorphisms, acting on X. Assume that R is a Dedekind complete Riesz space, and let $\overline{R} = R \cup \{\infty\}$, where ∞ satisfies properties (4.1.1).*

Let $p, q \colon X \to \overline{R}$ two maps, such that

(i) $p(0) = q(0) = 0$;
(ii) *q is monotonic (non-decreasing);*
(iii) *p is subadditive, and q is superadditive;*
(iv) *p is G-subinvariant, and q is G-superinvariant (that is, $p(gx) \leq p(x)$, $q(gx) \geq q(x)$, $\forall\, g \in G$, $\forall\, x \in X$);*
(v) $q(x) \leq p(x)$, $\forall x \in X$.

Then, there exists a monotone, additive, G-invariant map ψ, such that $q(x) \leq \psi(x) \leq p(x)$, $\forall\, x \in X$.

Before proving Theorem 4.2, we need some definitions and lemmas. Let Ω be such that R can be embedded as a solid subspace of $\mathcal{C}_\infty(\Omega)$, and let $\underline{m}$ be the function which associates the number $m \in \mathbb{N}$ with all $\omega \in \Omega$. Set $\mathcal{L} = \{f \in \overline{R}^G : f$ is positive, and $\sup\limits_{m \in \mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g) \in R\}$, where the considered integral is the C-integral.

Now, we see that $f_1 + f_2 \in \mathcal{L}$, whenever f_1, $f_2 \in \mathcal{L}$. Indeed, we have

$$\begin{aligned} 0 &\leq \big[f_1(g) + f_2(g)\big] \wedge \underline{m} \leq f_1(g) \wedge \underline{m} + f_2(g) \wedge \underline{m} \leq \\ &\leq \big[f_1(g) + f_2(g)\big] \wedge 2\underline{m}, \quad \forall\, g \in G,\ \forall\, f_1, f_2 \in \mathcal{L},\ \forall\, m \in \mathbb{N} \end{aligned}$$

(see also [12]). Taking the integrals, we obtain:

$$0 \leq \int_G [f_1(g) + f_2(g)] \wedge \underline{m}\, d\mu(g) \leq \int_G f_1(g) \wedge \underline{m}\, d\mu(g) + \int_G f_2(g) \wedge \underline{m}\, d\mu(g) \leq$$

$$\leq \sup_{m\in\mathbb{N}} \int_G f_1(g) \wedge \underline{m}\, d\mu(g) + \sup_{m\in\mathbb{N}} \int_G f_2(g) \wedge \underline{m}\, d\mu(g)\,;$$

taking the supremum, and by solidity of R in $\mathcal{C}_\infty(\Omega)$, we have that $f_1+f_2 \in \mathcal{L}$.

Lemma 4.3. *Let $f \in R^G$ be positive and bounded. Then $f \in \mathcal{L}$ Moreover,*

$$\int_G f(g)\, d\mu(g) = \sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g)\,.$$

Proof. By hypothesis, there exists $b \in R$ such that $0 \leq f(g) \leq b$, $\forall g \in G$. Thus, we have: $0 \leq \int_G f(g) \wedge \underline{m}\, d\mu(g) \leq b$, $\forall m \in \mathbb{N}$. By solidity of R, we obtain: $\int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$, $\forall m \in \mathbb{N}$. As R is Dedekind complete, it holds: $\sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$. The second assertion is a direct consequence of the construction of the C-integral. □

Now, we note that, if $f \in \overline{R}^G$ is bounded from below, then the map $g \mapsto [f(g)]^-$ is bounded. For every $f \in \mathcal{L}$, let $\overline{M}(f) = \sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g)$. It is easy to see that $\overline{M}(f_1+f_2) = \overline{M}(f_1) + \overline{M}(f_2)$, $\forall f_1, f_2 \in \mathcal{L}$.

Let us prove the following:

Lemma 4.4. *Let $f \in \overline{R}^G$, $h \in G$, and $f_h(g) = f(gh)$. Suppose that f is positive. Then, $f \in \mathcal{L}$ if and only if $f_h \in \mathcal{L}$.*

Proof. We prove that $[f_h \in \mathcal{L}] \Rightarrow [f \in \mathcal{L}]$: the converse implication is similar. As

$$0 \leq \int_G f(gh) \wedge \underline{m}\, d\mu(g) \leq \sup_{m\in\mathbb{N}} \int_G f(gh) \wedge \underline{m}\, d\mu(g) \in R,$$

then, by virtue of solidity of R in $\mathcal{C}_\infty(\Omega)$, it is:

$$\int_G f(gh) \wedge \underline{m}\, d\mu(g) \in R, \quad \forall m \in \mathbb{N}.$$

But

$$\int_G f(g) \wedge \underline{m}\, d\mu(g) = \int_G f(gh) \wedge \underline{m}\, d\mu(g), \quad \forall m \in \mathbb{N},$$

and hence $\int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$. As

$$\int_G f(g) \wedge \underline{m}\, d\mu(g) \leq \sup_{m\in\mathbb{N}} \int_G f(gh) \wedge \underline{m}\, d\mu(g) \in R,$$

the family $\{\int_G f(g) \wedge \underline{m}\, d\mu(g)\}_{m\in\mathbb{N}}$ is bounded from above in R and hence it has a supremum in R. So, $f \in \mathcal{L}$. It is easy to see that $\overline{M}(f_h) = \overline{M}(f)$, for all $f \in \mathcal{L}$, and for every $h \in G$. □

Now, set $\mathcal{H} = \{f \in R^G : f$ is positive and bounded $\}$, and let $\mathcal{I} = \mathcal{L} - \mathcal{H}$. We observe that $f_1 + f_2 \in \mathcal{I}$ whenever $f_1, f_2 \in \mathcal{I}$, and, if $f_1 \leq f_2$, $f_2 \in \mathcal{I}$ and f_1 is bounded from below, then $f_1 \in \mathcal{I}$. We define $M : \mathcal{I} \to R$ by setting

$$M(f) = \overline{M}(f_1) - \overline{M}(f_2) \quad \text{whenever} \quad f = f_1 - f_2, \quad \text{with} \quad f_1 \in \mathcal{L},\ f_2 \in \mathcal{H}. \tag{$*$}$$

One can check easily that M is well-defined, additive and monotone. For each $f \in \mathcal{I}$, let $\int_G f(g)\, d\mu(g) = M(f)$.

Now, we prove the following:

LEMMA 4.5. *Let $f \in \overline{R}^G$ be bounded from below. Then, $f \in \mathcal{I}$ if and only if $\sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$.*

Proof. Let $f \in \mathcal{I}$. Then there exist $f_1 \in \mathcal{L}$, $f_2 \in \mathcal{H}$, such that $f = f_1 - f_2$. Thus, $f_1(g) \geq f(g)$, $\forall g \in G$. As f is bounded from below, there exists $r \in R$ such that

$$f_1(g) \wedge \underline{m} \geq f(g) \wedge \underline{m} \geq r \wedge 0, \quad \forall g \in G, \forall m \in \mathbb{N}.$$

As $f_1 \in \mathcal{L}$, we have:

$$R \ni \sup_{m\in\mathbb{N}} \int_G f_1(g) \wedge \underline{m}\, d\mu(g) \geq \int_G f_1(g) \wedge \underline{m}\, d\mu(g) \geq$$
$$\geq \int_G f(g) \wedge \underline{m}\, d\mu(g) \geq r \wedge 0,$$

and thus $\int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$, $\forall m \in \mathbb{N}$. The family $\{\int_G f(g) \wedge \underline{m}\, d\mu(g)\}_{m\in\mathbb{N}}$ is bounded from above; then, by the Dedekind completeness of R, $\sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g) \in R$.

We prove the "if" part. By hypothesis, there exists $s \in R$ such that

$$f(g) = [f(g)]^+ - [f(g)]^- \geq [f(g)]^+ + s, \quad \forall g \in G,$$

and thus

$$f(g) \wedge m \geq [f(g)]^+ \wedge m + s \wedge 0, \quad \forall g \in G, \forall m \in \mathbb{N}.$$

So,

$$R \ni \sup_{m\in\mathbb{N}} \int_G f(g) \wedge \underline{m}\, d\mu(g) - s \wedge 0 \geq \int_G [f(g)]^+ \wedge \underline{m}\, d\mu(g) \geq 0, \quad \forall m \in \mathbb{N}.$$

The set $\{\int_G [f(g)]^+ \wedge \underline{m}\, d\mu(g)\}_{m\in\mathbb{N}}$ is a bounded subset of R, and hence $\sup_{m\in\mathbb{N}} \int_G [f(g)]^+ \wedge \underline{m}\, d\mu(g) \in R$. Thus, the map $g \mapsto [f(g)]^+$ belongs to $\mathcal{L}$. As the map $g \mapsto [f(g)]^-$ belongs to $\mathcal{H}$, then the assertion follows. □

Thanks to Lemma 4.5, and arguing as in Lemma 4.4, we can show the following:

Lemma 4.6. *Let $f \in \overline{R}^G$, f bounded from below. Then, $f \in \mathcal{I}$ if and only if $f_h \in \mathcal{I}$.*

It is easy to show that, if $f \in \mathcal{I}$, $M(f_h) = M(f)$, where M is defined as in $(*)$.

Now, we prove the following:

Lemma 4.7. *Assume that $f_1, f_2 \in \overline{R}^G$ are bounded from below, and suppose that $f_1 + f_2 \in \mathcal{I}$. Then $f_1 \in \mathcal{I}$ and $f_2 \in \mathcal{I}$.*

Proof. It will be enough to prove that $f_1 \in \mathcal{I}$. We show that $\sup_{n\in\mathbb{N}} \int_G f_1(g) \wedge \underline{n}\, d\mu(g) \in R$: then, by Lemma 4.5, we will have that $f_1 \in \mathcal{I}$.

As f_1, f_2 are bounded from below, there exist $w_1, w_2 \in R$ such that $f_1(g) \geq w_1$, $f_2(g) \geq w_2$, $\forall g \in G$. Then, for each fixed $n, p \in \mathbb{N}$, we find:

$$w_1 \wedge 0 + w_2 \wedge 0 \leq f_1(g) \wedge \underline{n} + f_2(g) \wedge \underline{p} \leq$$
$$\leq f_1(g) \wedge \underline{m} + f_2(g) \wedge \underline{m} \leq [f_1(g) + f_2(g)] \wedge 2\underline{m},$$

where $m = \max(n, p)$. Taking the integrals, we obtain:

$$w_1 \wedge 0 + w_2 \wedge 0 \leq \int_G f_1(g) \wedge \underline{n}\, d\mu(g) + \int_G f_2(g) \wedge \underline{p}\, d\mu(g) \leq$$
$$\leq \sup_{m\in\mathbb{N}} \int_G [f_1(g) + f_2(g)] \wedge \underline{m}\, d\mu(g);$$

by hypothesis and Lemma 4.5, the last quantity exists in R. So, we have

$$w_1 \wedge 0 \leq \int_G f_1(g) \wedge \underline{n}\, d\mu(g) \leq \sup_m \int_G [f_1(g) + f_2(g)] \wedge \underline{m}\, d\mu(g) -$$
$$- \int_G f_2(g) \wedge \underline{p}\, d\mu(g) \leq \sup_m \int_G [f_1(g) + f_2(g)] \wedge \underline{m}\, d\mu(g) - w_2 \wedge 0.$$

By solidity of R in $\mathcal{C}_\infty(\Omega)$, we see that $\int_G f_1(g) \wedge \underline{n}\, d\mu(g) \in R$, for all $n \in \mathbb{N}$. The family $\{\int_G f_1(g) \wedge \underline{n}\, d\mu(g)\}_{n\in\mathbb{N}}$ is bounded from above; thus, by the Dedekind completeness of R, $\sup_{n\in\mathbb{N}} \int_G f_1(g) \wedge \underline{n}\, d\mu(g) \in R$. □

Now, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. By a well-known result by Fuchssteiner and Lusky ([7]), there exists a monotone, additive map $\varphi\colon X \to \overline{R}$, such that $q \le \varphi \le p$.

Now, for each fixed $x \in X$, define $f_x\colon G \to \overline{R}$ by setting $f(x) = \varphi(gx)$, and construct a mapping $\psi\colon X \to \overline{R}$, in the following way.
If $q(x) = \infty$, put $\psi(x) = \infty$.
If $q(x) \ne \infty$, set

$$\psi(x) = \begin{cases} \int\limits_G f_x(g)\, d\mu(g), & \text{if } f_x \in \mathcal{I}, \\ \infty, & \text{if } f_x \notin \mathcal{I}. \end{cases}$$

First of all, we prove that $q(x) \le \psi(x) \le p(x)$, for all $x \in X$: it is enough to show it, assuming $q(x) \ne \infty$ and $p(x) \ne \infty$. As $q(x) \le q(gx) \le \varphi(gx) \le p(gx) \le p(x)$, $\forall x \in X$, $\forall g \in G$, we see that f_x is bounded, and thus $f_x \in \mathcal{I}$; hence, $q(x) \le \int\limits_G f_x(g)\, d\mu(g) \le p(x)$.

Now, we show that $\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)$, for every fixed $x_1, x_2 \in X$. We first consider the case $q(x_1 + x_2) \ne \infty$, $q(x_1) \ne \infty$, $q(x_2) \ne \infty$. By Lemma 4.7, we have that $f_{x_1+x_2} \in \mathcal{I}$ if and only if $f_{x_1} \in \mathcal{I}$ and $f_{x_2} \in \mathcal{I}$: indeed, it is easy to see that $f_{x_1+x_2} \equiv f_{x_1} + f_{x_2}$. In the case $f_{x_1}, f_{x_2}, f_{x_1+x_2} \in \mathcal{I}$, one has:

$$\psi(x_1 + x_2) = \int\limits_G f_{x_1+x_2}(g)\, d\mu(g) = \int\limits_G f_{x_1}(g)\, d\mu(g) + \int\limits_G f_{x_2}(g)\, d\mu(g) = \\ = \psi(x_1) + \psi(x_2).$$

Otherwise, it is: $\psi(x_1 + x_2) = \infty = \psi(x_1) + \psi(x_2)$. Now, let $q(x_1) = \infty$ or $q(x_2) = \infty$. Then, $q(x_1 + x_2) \ge q(x_1) + q(x_2)$ implies that $q(x_1 + x_2) = \infty$, and $\psi(x_1 + x_2) = \infty = \psi(x_1) + \psi(x_2)$. Finally, let $q(x_1 + x_2) = \infty$, $q(x_1) \ne \infty$, $q(x_2) \ne \infty$. By superinvariance of q, we have $q\bigl(g(x_1+x_2)\bigr) = q(gx_1 + gx_2) = \infty$, $\forall g \in G$. But φ is additive, so: $\infty = \varphi(gx_1 + gx_2) = \varphi(gx_1) + \varphi(gx_2)$, $\forall g \in G$. Thus, $\varphi(gx_1) = \infty$, or $\varphi(gx_2) = \infty$.

Let $A_{x_1,x_2} = \{g \in G\colon \varphi(gx_1) = \infty\}$, $B_{x_1,x_2} = \{g \in G\colon \varphi(gx_2) = \infty\}$. Then, $G = A_{x_1,x_2} \cup B_{x_1,x_2}$. Hence, $\mu(A_{x_1,x_2}) > 0$, or $\mu(B_{x_1,x_2}) > 0$. Thus, $f_{x_1} \notin \mathcal{I}$, or $f_{x_2} \notin \mathcal{I}$, and so $\psi(x_1) = \infty$, or $\psi(x_2) = \infty$. So, we get: $\psi(x_1) + \psi(x_2) = \infty = \psi(x_1 + x_2)$.

Now, we prove that ψ is invariant, that is $\psi(hx) = \psi(x)$, $\forall x \in X$, $\forall h \in G$. If $q(x) = \infty$, then $q(hx) = \infty$, and hence $\psi(x) = \psi(hx) = \infty$. Let $q(x) \ne \infty$, $q(hx) = \infty$. Then

$$f_{x,h}(g) = f_x(gh) = \varphi(ghx) \ge q(ghx) \ge q(hx) = \infty, \quad \forall g \in G.$$

Thus, $f_{x,h} \notin \mathcal{I}$. By Lemma 4.6, $f_x \notin \mathcal{I}$, and hence $\psi(x) = \infty = \psi(hx)$. Finally, let $q(x) \ne \infty$, $q(hx) \ne \infty$. If $f_x \notin \mathcal{I}$, we have $f_{x,h} \notin \mathcal{I}$, and thus

$\psi(x) = \psi(hx) = \infty$. If $f_x \in \mathcal{I}$, then $f_{x,h} \in \mathcal{I}$, and we have:

$$\psi(hx) = \int_G \varphi(ghx)\, d\mu(g) = \int_G \varphi(gx)\, d\mu(g) = \psi(x)\,.$$

So, ψ is invariant.

Finally, we show that ψ is monotone. Let $x_1, x_2 \in X$, $x_1 \le x_2$. By hypothesis, $q(x_1) \le q(x_2)$. If $q(x_2) = \infty$, then we have $\psi(x_2) = \infty \ge \psi(x_1)$. If $q(x_2) \ne \infty$, then $q(x_1) \ne \infty$. Moreover, $f_{x_1} \le f_{x_2}$, by virtue of monotonicity of φ and of the homomorphisms $g \in G$. If $f_{x_2} \notin \mathcal{I}$, one has $\psi(x_2) = \infty \ge \psi(x_1)$. If $f_{x_2} \in \mathcal{I}$, then $f_{x_1} \in \mathcal{I}$: in this case, we obtain:

$$\psi(x_1) = \int_G \varphi(gx_1)\, d\mu(g) \le \int_G \varphi(gx_2)\, d\mu(g) = \psi(x_2)\,.$$

Thus, ψ is monotone. This concludes the proof of Theorem 4.2. □

R e m a r k 4.8. In Theorem 4.2, assume in addition that X is a preordered cone, and q satisfies the following condition:

$$\limsup_{\varepsilon \downarrow 0}\ q(\varepsilon x) = 0, \quad \forall\, x \in X\,; \tag{4.8.1}$$

then the map ψ turns out to be linear (see [7]).

DEFINITION 4.9. A map $q\colon X \to R$ is said to be *superlinear* if it is superadditive and $q(\lambda\, x) = \lambda\, q(x)$, $\forall\, x \in X$, $\forall\, \lambda \in \mathbb{R}_0^+$.

The following theorem is a consequence of Remark 4.8.

THEOREM 4.10. *Under the same hypotheses as in Theorem 4.2, suppose that X is a preordered cone, and q is a superlinear map. Then the mapping ψ in Theorem 4.2 is linear.*

DEFINITION 4.11. We will say that a semigroup G (not necessarily amenable) *satisfies property* [S] with respect to a Dedekind complete Riesz space R if it satisfies Theorem 4.2, [Theorem (4.10)] for R and for all X, p, q as in the hypothesis of Theorem 4.2, [Theorem (4.10)].

DEFINITION 4.12. We say that a semigroup G *satisfies property* [H-B] with respect to R if, for each preordered Abelian group [vector space] X on which G acts, for all invariant subvariety $Z \subset X$, for every monotone, subadditive [sublinear], G-subinvariant map $p\colon X \to R$, for any monotone, additive [linear], G-invariant functional $\mu\colon Z \to R$, such that $\mu \le p$, then there exists an invariant extension $\tilde{\mu}$ of μ, $\tilde{\mu}\colon X \to R$, such that $\tilde{\mu} \le p$.

DEFINITION 4.13. We will say that a semigroup G *satisfies property* [E] with respect to R if, for all X, Z, μ as in Definition 4.12, with Z cofinal in X, then there exists an invariant extension of μ, $\hat{\mu}\colon X \to R$.

THEOREM 4.14. *Let G be a semigroup, and let R be a fixed Dedekind complete Riesz space. Then, the following statements are equivalent:*

(4.14.1) *G is right-amenable,*
(4.14.2) *G satisfies* [S] *with respect to R,*
(4.14.3) *G satisfies* [H-B] *with respect to R,*
(4.14.4) *G satisfies* [E] *with respect to R.*

Proof.

(4.14.1) $\Rightarrow$ (4.14.2) Theorem 4.2, [Theorem (4.10)].

(4.14.2) $\Rightarrow$ (4.14.3) See [1].

(4.14.3) $\Rightarrow$ (4.14.4) See [2].

(4.14.4) $\Rightarrow$ (4.14.1) We note that (4.13.4) implies the existence of a finitely additive, monotone, G-invariant map, such that $\mu(G)$ is a unit u of R: indeed, one can choose $X = \{f \in \mathbb{R}^G,\, f$ is bounded$\}$, $Z = \{f \in X,\, f$ is constant$\}$, $g \in X^X$ defined by setting: $g(f)(h) = f(gh)$, $\forall f \in X$, $\forall h \in G$; $\mu(\underline{m}) = mu$, if $\underline{m}$ is the constant function, whose value is m. By (4.14.4), the map μ has an extension $\tilde{\mu}$. Taking $\hat{\mu}(A) = \tilde{\mu}(\chi_A)$, $\forall A \subset G$, we have that $\hat{\mu}$ is the required map.

Now, we prove amenability of G. Let Ω be such that R can be embedded as a solid subspace in $\mathcal{C}_\infty(\Omega)$. Then: $0 \le \hat{\mu}(A) \le \hat{\mu}(G)$, $\forall A \subset G$, and hence there exists $S \subset \Omega$, S open and dense, such that $\hat{\mu}(A)$ is real on S, $\forall A \subset G$. As $\hat{\mu}(G) = u$ is a unit, there exists $t \in S$ such that $\hat{\mu}(G)(t) > 0$.

Put $\nu(A) = \frac{\hat{\mu}(A)(t)}{\hat{\mu}(G)(t)}$, for each $A \subset G$. It is easy to see that $\nu(\emptyset) = 0$ and $\nu(G) = 1$. If $A, B \subset G$, $A \cap B = \emptyset$, we have

$$\nu(A \cup B) = \frac{\hat{\mu}(A \cup B)(t)}{\hat{\mu}(G)(t)} = \frac{\hat{\mu}(A)(t) + \hat{\mu}(B)(t)}{\hat{\mu}(G)(t)} = \nu(A) + \nu(B).$$

It is easy to check that ν is $[0,1]$-valued and right-invariant. Thus, G is right-amenable.

Now, given any set H, write $L^\infty(H) = \{f \in \mathbb{R}^H :\ f$ is bounded$\}$. By means of the π-integral, it is possible to prove the following well-known result (see also [16]).

THEOREM 4.15. *Let Ω be a compact Stonian topological space, such that $\mathcal{C}(\Omega)$ is a π-space. Assume that H is a right-amenable group of homeomorphisms $h \in \Omega^\Omega$, and suppose that:*

(4.15.1) *There exists $\omega_0 \in \Omega$ such that the set $\{g(\omega_0) \colon g \in H\}$ is dense in Ω. Then there exists a linear monotone map $\psi \colon L^\infty(H) \to \mathcal{C}(\Omega)$, such that $\psi(z) = z$, $\forall z \in \mathcal{C}(\Omega)$, and $\psi(hx) = h\psi(x)$, $\forall h \in H$, $\forall x \in L^\infty(H)$.*

In the literature, the maps satisfying Theorem 4.15. are called *H-equivariant.*

REFERENCES

[1] BOCCUTO, A.—CANDELORO, D.: *Sandwich theorems and applications to invariant measures*, Atti Sem. Mat. Fis. Univ. Modena **38** (1990), 511–524.

[2] BOCCUTO, A.—CANDELORO, D.: *Sandwich theorems, extension principles and amenability,* (1992).

[3] CHOJNACKI, W.: *Sur un théorème de Day, un théorème de Mazur-Orlicz et une généralisation de quelques théorèmes de Silverman*, Colloq. Math. **50** (1986), 257–262.

[4] FILTER, W.: *Dual spaces of $C_\infty(X)$*, Rend. Circ. Mat. Palermo, Serie II **35** (1986), 135–158.

[5] FILTER, W.: *Representations of Archimedean Riesz spaces – a survey,* (1990).

[6] FREMLIN, D. H.: *Topological Riesz Spaces and Measure Theory*, Proc. London Math. Soc. Cambridge Univ. Press., 1974.

[7] FUCHSSTEINER, B—LUSKY, W.: *Convex cones*, North–Holland Publishing Co. (1981).

[8] GLEASON, A. M.: *Projective topological spaces Ill.*, J. Math. **2** (1958), 482–489.

[9] LUXEMBURG, W. A. J.—MASTERSON, J. J.: *An extension of the concept of the order dual of a Riesz space*, Canad. J. Math. **26** (1965), 488–498.

[10] LUXEMBURG, W. A. J.—ZAANEN, A. C.: *Riesz spaces, vol. I*, North–Holland Publishing Co. (1971).

[11] McGILL, P.: *Integration in vector lattices*, J. London Math. Soc. **11** (1975), 347–360.

[12] NAMIOKA, I.: *Partially ordered linear topological spaces*, Mem. Amer. Math. Soc. **24** (1957).

[13] ORIHARA, M.—SUNOUCHI, G.: *An abstract integral VIII*, Proc. Imp. Acad. Tokyo **18** (1942), 535–538.

[14] RAINWATER, J.: *A note on projective resolutions*, Proc. Amer. Math. Soc. **10** (1959), 734–735.

[15] RIEČAN, B.: *On the Kurzweil integral for functions with values in ordered spaces I*, Acta Math. Univ. Comenian. **56–57** (1989), 411–424.

[16] WRIGHT, J. D. M.: *Paradoxical decompositions of the cube and injectivity*, Bull. London Math. Soc. **22** (1990), 18–24.

Received March 17, 1993

Department of Mathematics
Via Vanvitelli, 1
I–06123 Perugia
ITALY
E-mail: tipo@ipguniv.bitnet