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HEWITT–YOSIDA DECOMPOSITION FOR □–DECOMPOSABLE MEASURES

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ABSTRACT. In the present paper, □-decomposable measures defined on a σ-complete lattice with relative complement and with values in a σ-complete lattice ordered semigroup are considered. For such measures a Hewitt–Yosida decomposition theorem is proved.

1. Introduction

Klement and Weber [6] gave a unified approach to several concepts of measures introducing the notion of generalized measure. Let $(\mathbf{L}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be a σ-complete lattice with smallest and largest element $\mathbf{0}$ and $\mathbf{1}$, respectively, and let $(\mathbf{S}, \square, \leq, 0, 1)$ be a σ-complete, lattice ordered commutative semigroup with identity 0 and with the smallest and largest element 0 and 1, respectively.

DEFINITION 1. A mapping $m : \mathbf{L} \to \mathbf{S}$ satisfying

$$m(\mathbf{0}) = 0\,,$$

$$m(x \wedge y) \,\square\, m(x \vee y) = m(x) \,\square\, m(y)\,,$$

$$(x_n)_{n \in \mathrm{N}} \uparrow \Rightarrow \sup_{n \in \mathrm{N}} m(x_n) = m\Bigl(\bigvee_{n \in \mathrm{N}} x_n\Bigr)\,,$$

is called an $\mathbf{S}$-*valued measure* on $\mathbf{L}$.

It turns out that this notion is very useful as a unified approach to several concepts of measures: σ-additive measure, probability measures on fuzzy events [17], possibility measures [18], fuzzy probability measures [5], fuzzy-valued fuzzy measures [6], σ-$\perp$-decomposable measures [14] and [9], measures on fuzzy events [6],

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$\oplus$-decomposable measures [6], S t o n e and W*-algebra-valued positive measures [15, 16]. An $\mathbf{S}$-valued measure m has the following property

$$m(x \vee y) = m(x) \square m(y) \quad \text{for} \quad x \wedge y = \mathbf{0},$$

i.e., m is a $\square$-decomposable measure.

We shall prove in this paper a Hewitt–Yosida type theorem for $\square$-decomposable measures continuous from above and with additional suppositions on the domain and on the range.

2. The set $D_{\downarrow}(\mathbf{L}, \mathbf{S})$

Definition 2. A lattice $\mathbb{L}$ is called a lattice with a *relative complement* if, for each element x from any interval $[a, b]$, there exits an element y such that

$$x \vee y = b \quad \text{and} \quad x \wedge y = a.$$

The element y is called the relative complement of the element x on the interval $[a, b]$.

Definition 3. A lattice $\mathbb{L}$ is called a *sectionally complemented lattice* if, for each element x from any interval $[\mathbf{0}, b]$, there exits an element y such that

$$x \vee y = b \quad \text{and} \quad x \wedge y = \mathbf{0}.$$

R e m a r k 1. The complement, in general, is not unique. For distributive lattices with relative complement the complement is unique for each element. So for Boolean algebras the complement always exists and it is unique.

We shall suppose in the whole paper that $\mathbb{L}$ is a σ-complete, sectionally complemented lattice.

Theorem 1. *If $\{x_n\}$ is a sequence from $\mathbb{L}$ such that $x_n \downarrow x$, then*

$$\inf_n y_n = \mathbf{0},$$

where y_n is a relative complement of x on $[\mathbf{0}, x_n]$, in addition, there exists a sequence $\{y_n\}$ such that $y_n \not< y_{n+1}$ $(n \in \mathbb{N})$.
If $\mathbb{L}$ is a distributive lattice, then $y_n \downarrow \mathbf{0}$.

P r o o f. Let y_n be a relative complement of x on $[\mathbf{0}, x_n]$ $(n \in \mathbb{N})$, i.e., $x \vee y_n = x_n$ and $x \wedge y_n = \mathbf{0}$. Since $x_n \downarrow x$, we have $\inf_n x_n = x$ and so that

$x = \inf_n (x \vee y_n)$. $\mathbb{L}$ is a σ-complete lattice and $y_n \geq \mathbf{0}$, so $\inf_n y_n = y$ always exists. We have

$$x = \inf_n (x \vee y_n) \geq x \vee \inf_n y_n = x \vee y\,,$$

i.e., $x \geq y$. Since $x \wedge y_n = \mathbf{0}$; $(n \in \mathbb{N})$ holds, we obtain

$$\inf_n (x \wedge y_n) = \mathbf{0}\,.$$

Hence

$$x \wedge y = x \wedge \inf_n y_n = \inf_n (x \wedge y_n) = \mathbf{0}\,.$$

Since $x \geq y$, we obtain $y = \mathbf{0}$. Now, let $\mathbb{L}$ be a distributive lattice. Then we have

$$x \vee (y_n \wedge y_{n+1}) = (x \vee y_n) \wedge (x \vee y_{n+1}) = x_n \wedge x_{n+1} = x_{n+1}\,.$$

Hence by

$$x \wedge (y_n \wedge y_{n+1}) = \mathbf{0}\,,$$

we obtain that $y_n \wedge y_{n+1}$ is also a relative complement of x on $[\mathbf{0}, x_{n+1}]$. Since $\mathbb{L}$ is a distributive lattice, the relative complement is unique and so we have

$$y_{n+1} = y_{n+1} \wedge y_n \leq y_n\,.$$

□

Definition 4. A lattice semigroup $\mathbf{S}$ is lower complete if every majorised increasing net (x_α) in $\mathbf{S}$ has the least upper bound, i.e., $\bigvee_\alpha x_\alpha \in \mathbf{S}$.

We suppose in the whole paper that $\mathbf{S}$ is a lower complete lattice and that

$$\inf(A \,\Box\, x) = (\inf A) \,\Box\, x \quad (A \subset \mathbf{S},\ x \in \mathbf{S}) \tag{$*$}$$

holds.

Definition 5. A mapping $m : \mathbb{L} \to \mathbf{S}$ satisfying

$$m(\mathbf{0}) = 0\,,$$

$$m(x \vee y) = m(x) \,\Box\, m(y)$$

for $x, y \in \mathbb{L}$ such that $x \wedge y = \mathbf{0}$, is called a *□-decomposable measure on* $\mathbb{L}$.

THEOREM 2. *Let m, $m : \mathbf{L} \to \mathbf{S}$, be a non-trivial $\Box$-decomposable measure in the sense that $y \,\Box\, v = y$ for each $y \in m(\mathbf{L})$, $y \neq \mathbf{0}$, and $v \in m(\mathbf{L})$ implies $v = 0$. Then m is continuous from above, i.e., $x_n \downarrow x$ implies*

$$\inf_n m(x_n) = m(x)\,,$$

iff

$$\inf_n m(y_n) = 0\,,$$

where y_n is a relative complement of x on $[\mathbf{0}, x_n]$.

Proof. Let $\{x_n\}$ be a sequence from $\mathbf{S}$ such that $x_n \downarrow x$. Let y_n be a relative complement of x on $[\mathbf{0}, x_n]$, i.e.,

$$x \vee y_n = x_n \quad \text{and} \quad x \wedge y_n = \mathbf{0}\,.$$

We suppose

$$\inf_n m(y_n) = 0\,.$$

Then we obtain

$$\inf_n m(x_n) = \inf_n m(y_n \vee x) = \inf_n m(y_n) \,\Box\, m(x) = m(x)\,.$$

Suppose now that $x_n \downarrow x$ implies $\inf_n m(x_n) = m(x)$. If y_n $(n \in \mathbb{N})$ is a relative complement of x on $[\mathbf{0}, x_n]$, then we have

$$m(x) = \inf_n m(x_n) = \inf_n m(x \vee y_n) = \inf_n m(y_n) \,\Box\, m(x)\,.$$

Hence $\inf_n m(y_n) = 0$. $\Box$

The set of all non-trivial (in the sense of Theorem 2) $\Box$-decomposable mappings on $\mathbf{L}$ into $\mathbf{S}$ will be denoted by $D(\mathbf{L}, \mathbf{S})$ and their subset of all continuous from above mappings will be denoted by $D_\downarrow(\mathbf{L}, \mathbf{S})$. We endow the set $D(\mathbf{L}, \mathbf{S})$ with the usual pointwise order, i.e., for $m_1, m_2 \in D(\mathbf{L}, \mathbf{S})$

$$m_1 \le m_2 \quad \text{iff} \quad m_1(x) \le m_2(x) \quad (x \in \mathbf{L})\,,$$

and with the operation $\Box$ defined by

$$(m_1 \,\Box\, m_2)(x) = m_1(x) \,\Box\, m_2(x) \quad (x \in \mathbf{L})\,.$$

EXAMPLE 1. Let $\perp$ be an Archimedean t-conorm, i.e., a function $\perp: [0,1] \times [0,1] \to [0,1]$ which is nondecreasing in each argument, commutative, associative, 0 is the unit element, continuous and $\perp(x,x) > x$ for all $x \in (0,1)$. Then by S. Weber [14], taking $\mathbf{S} = [0,1]$, $\Box = \perp$ and $\mathbf{L} = \Sigma$ a σ-algebra of subsets of a set Ω, we obtain that $D(\Sigma, [0,1])$ is the set of $\perp$-decomposable measures and that $D_{\downarrow}(\Sigma, [0,1])$ is a subset of the set $D_{\sigma}(\Sigma, [0,1])$ of σ-$\perp$-decomposable measures. For the case (NSA) (see for the notation [14])

$$D_{\downarrow}(\Sigma, [0,1]) = D_{\sigma}(\Sigma, [0,1]) .$$

EXAMPLE 2. Let $\perp$ be the t-conorm max on $[0,1]$. Then the elements of $D_{\downarrow}(\Sigma, [0,1])$ are continuous from below and so

$$D_{\downarrow}(\Sigma, [0,1]) \subset D_{\sigma}(\Sigma, [0,1]) .$$

3. Hewitt–Yosida decomposition

We suppose in this section that $\mathbf{S}$ satisfies $(*)$ and also the following conditions:

(a) it is of the countable type, i.e., every subset A of $\mathbf{L}$ that has a supremum in $\mathbf{L}$, contains a countable subset A_1 such that $\sup A = \sup A_1$.

(b) $x \Box \sup A = \sup(x \Box A)$, whenever there exist suprema.

(c) If $a_{in} \in \mathbf{S}$ such that $\inf\limits_n a_{in} = 0$ $(i \in \mathbb{N})$ and $a_{in} \not< a_{i(n+1)}$ $(i, n \in \mathbb{N})$, then

$$\inf_n \sup_i a_{in} = 0 ;$$

or instead of (c)

(c_1) $\mathbf{L}$ is a distributive lattice and if $a_{in} \in \mathbf{S}$ such that $a_{in} \downarrow 0$ $(i \in \mathbb{N})$, then

$$\inf_n \sup_i a_{in} = 0 .$$

Remark 2. Similar conditions as (c_1) can be find in papers of B. Riečan [12] and D. Maharam Stone [7].

DEFINITION 6. A non-empty subset S_1 of $\mathbf{S}$ is a *band* of $\mathbf{S}$ if it satisfies the following conditions:

(i) $x, y \in S_1$ implies $x \Box y \in S_1$;

(ii) $x \leq y$ and $y \in S_1$ imply $x \in S_1$;

(iii) For any increasing net in S_1 its least upper bound (if it exists) belongs to S_1.

THEOREM 3. *Let $D(\mathbb{L}, \mathbf{S})$ be a lattice. Then*
$D_{\downarrow}(\mathbb{L}, \mathbf{S})$ is a band of the lower complete lattice semigroup $D(\mathbb{L}, \mathbf{S})$.

P r o o f . We shall prove that $D(\mathbb{L}, \mathbf{S})$ is a lower complete lattice semigroup. Let $(m_i)_{i \in I}$ be an increasing net in $D(\mathbb{L}, \mathbf{S})$ such that for every x there is $b \in \mathbf{S}$ such that $m_i(x) \leq b$ and

$$m = \sup_{i \in I} m_i, \quad \text{i.e.,} \quad m(x) = \sup_i m_i(x) \ (x \in \mathbb{L}).$$

Then for all $x, y \in \mathbb{L}$ such that $x \wedge y = \mathbf{0}$ we have

$$m(x \vee y) = \sup_i m_i(x \vee y) = \sup_i (m_i(x) \,\Box\, m_i(y)) \leq \sup_i m_i(x) \,\Box\, \sup_i m_i(y).$$

Let i, j be any pair of indices from I. There exists $k \in I$ such that $i < k$ and $j < k$. Then we have

$$m_i(x) \,\Box\, m_j(y) \leq m_k(x) \,\Box\, m_k(y) = m_k(x \vee y) \leq m(x \vee y).$$

Hence by the first inequality

$$m(x) \,\Box\, m(y) = m(x \vee y).$$

We shall prove that $D_{\downarrow}(\mathbb{L}, \mathbf{S})$ is a band of $D(\mathbb{L}, \mathbf{S})$. It is obvious that $m_1, m_2 \in D_{\downarrow}(\mathbb{L}, \mathbf{S})$ implies

$$m_1 \,\Box\, m_2 \in D_{\downarrow}(\mathbb{L}, \mathbf{S}).$$

Suppose $m \in D(\mathbb{L}, \mathbf{S})$ and $\mu \in D_{\downarrow}(\mathbb{L}, \mathbf{S})$ such that $m \leq \mu$. Let $\{x_n\}$ be a sequence in $\mathbb{L}$ such that $x_n \downarrow x$, and y_n a relative complement of x on $[\mathbf{0}, x_n]$, then by Theorem 1, $\inf_n y_n = \mathbf{0}$ and since $\mu \in D_{\downarrow}(\mathbb{L}, \mathbf{S})$ by Theorem 2,

$$\inf_n \mu(y_n) = 0.$$

Hence

$$\inf_n m(y_n) = 0.$$

Then by Theorem 2, $m(x_n) \downarrow m(x)$, i.e., $m \in D_{\downarrow}(\mathbb{L}, \mathbf{S})$. Let $(m_i)_{i \in I}$ be an increasing net in $D_{\downarrow}(\mathbb{L}, \mathbf{S})$ and

$$m = \sup_{i \in I} m_i \in D(\mathbb{L}, \mathbf{S}).$$

Let $x_n \downarrow x$. Since $m_i \in D_{\downarrow}(\mathbb{L}, \mathbf{S})$ we have by Theorem 2,

$$\inf_n m_i(y_n) = 0,$$

where y_n is a relative complement of x on $[\mathbf{0}, x_n]$. Now we have by properties (a) and (c) or (c_1) (if $\mathbf{L}$ is a distributive lattice), using Theorem 1,

$$\inf_n m(y_n) = \inf_n \sup_i m_i(y_n) = \inf_n \sup_{i\in C} m_i(y_n) = 0\,,$$

where C is a countable subset of I. Using properties $(*)$ and (b) we obtain

$$\inf_n \sup_i m_i(x_n) = \inf_n \sup_{i\in C} m_i(x \vee y_n) = \inf_n \sup_{i\in C}(m_i(x) \,\Box\, m_i(y_n)) \le$$

$$\inf_n \sup_{i\in C} m_i(x) \,\Box\, \inf_n \sup_{i\in C} m_i(y_n) = \inf_n \sup_i m_i(x) = \inf_n m(x) = m(x)\,.$$

Hence by monotonicity of m_i, $m \in D_{\downarrow}(\mathbf{L}, \mathbf{S})$. □

We suppose further that the operation □ and the partial ordering $\le$ in $\mathbf{S}$ satisfy the conditions:

(i) $x, y \in \mathbf{S}$, $x \,\Box\, y = x \,\Box\, z$ imply $y = z$;
(ii) $x, y \in \mathbf{S}$, $x \,\Box\, y = 0$ imply $x = y = 0$;
(iii) $u \le v$ if there exists an element $w \in \mathbf{S}$ such that $v = u \,\Box\, w$;
(iv) if $x_\alpha \uparrow$, then

$$\sup_\alpha (x_\alpha \wedge y) = (\sup_\alpha x_\alpha) \wedge y\,.$$

Remark 3. By Nakada's theorem [4], conditions (i) and (iii) imply that $\mathbf{S}$ is a positive cone of an p. o. group.

Definition 7. For $S_1 \subset \mathbf{S}$, $S_1 \ne \emptyset$, we define

$$S_1^\perp = \{x : x \in \mathbf{S},\ x \wedge y = \mathbf{0} \quad \text{for every} \quad y \in S_1\}\,.$$

Lemma 1. *$S_1^\perp$ is a band of $\mathbf{S}$.*

Theorem 4. ([2], [11]) *If S_1 is a band of a lower complete lattice semigroup $\mathbf{S}$, then for any $u \in \mathbf{S}$, there exist unique $u' \in S_1$ and $u'' \in S_1^\perp$ such that*

$$u = u' \,\Box\, u''\,,$$

and

$$u' = \sup\{v : v \in S_1,\ 0 \le v \le u\}\,,$$
$$u'' = \sup\{v : v \in S_1^\perp,\ 0 \le v \le u\}\,.$$

Theorem 5. (*Hewitt–Yosida decomposition*) *Let $D(\mathbf{L}, \mathbf{S})$ be a lattice. For every $m \in D(\mathbf{L}, \mathbf{S})$, there exist unique $m_1 \in D_{\downarrow}(\mathbf{L}, \mathbf{S})$ and $m_2 \in D_{\downarrow}(\mathbf{L}, \mathbf{S})^\perp$ such that*

$$m = m_1 \,\Box\, m_2\,.$$

Proof. By Theorem 3, $D_{\downarrow}(\mathbf{L}, \mathbf{S})$ is a band in $D(\mathbf{L}, \mathbf{S})$, then by the preceding Theorem 4, it follows the desired decomposition. □

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