Tatra Mountains Math. Publ. **3** (1993), 77–80

TATRA MOUNTAINS
Mathematical Publications

FINITE INDEFINITE MEASURES ON HYPERBOLIC LOGICS

MARJAN S. MATVEJCHUK

ABSTRACT. We present a generalization of Gleason's theorem for finite indefinite measures on hyperbolic logics related with W^*–factors in a Krein space.

Let H be a space with an indefinite metric $[\cdot,\cdot]$, a canonical decomposition $H = H^+ [\dot{+}] H^-$, and with a canonical symmetry J. H is a Hilbert space with respect to the inner product $(x,z) = [Jx,z]$. There exist two orthogonal projections P^+ and P^- such that $P^+ + P^- = I$, $J = P^+ - P^-$ and $P^+H = H^+$, $P^-H = H^-$, $[x,z] = (Jx,z)$, for any $x, z \in H$. A W^*–factor $\mathcal{A}$ in H is called a *W^*J–factor*, if $J \in \mathcal{A}$. A W^*J–factor $\mathcal{A}$ is said to be a *$W^*\Pi$–factor* if at least one of projections P^+ or P^- is finite relative to $\mathcal{A}$. Let $\mathcal{A}^\Pi$ be the set of all orthogonal projections in $\mathcal{A}$ and Π $(= \Pi(\mathcal{A}))$ be the set of all J–selfadjoint projections in $\mathcal{A}$, i.e., $\Pi = \{p \in \mathcal{A}\colon p^2 = p,\ [px,z] = [x,pz]$, for any $x,z \in H\}$. Let Π_f be the set of all projections $p \in \Pi$ such that the subspace pH is finite relative to $\mathcal{A}$. Now let Π^+ (Π^-) be the set of all projections $p \in \Pi$, for which the subspace pH is positive $(\forall x \in pH,\ x \neq 0,\ [x,x] > 0)$ (respectively, negative, i.e., $\forall x \in pH,\ x \neq 0,\ [x,x] < 0)$. Any projection $e \in \Pi$ is representable in the form $e = e_+ + e_-$, where $e_+ \in \Pi^+$, $e_- \in \Pi^-$.

A mapping $\mu\colon \Pi \to \mathbb{R}$ $(\mu\colon \Pi_f \to \mathbb{R})$ is called a *measure* if, for any representation $e = \sum_\iota e_\iota$ (the sum is understood in the strong topology), we have $\mu(e) = \sum_\iota \mu(e_\iota)$. A measure μ is said to be a *semiconstant* if $\mu(e) = c\tau(e_+)$ $\forall e \in \Pi$ or $\mu(e) = c\tau(e_-)$ $\forall e \in \Pi$, where τ is a faithful normal semifinite trace on $\mathcal{A}$; *indefinite*, if $\mu|\Pi^+ \geq 0$ and $\mu|\Pi^- \leq 0$ $(\mu|\Pi^+ \cap \Pi_f \geq 0$ and $\mu|\Pi^- \cap \Pi_f \leq 0$, respectively). An indefinite measure $\mu\colon \Pi_f \to \mathbb{R}$ is said to be *finite* if, for any projection $p \in (\Pi^+ \cup \Pi^-)$, the inequality $\sup\{|\mu(q)|\colon q \leq p,\ q \in \Pi_f\} < \infty$ holds.

AMS Subject Classification (1991): 46G12, 46N50.
Key words: hyperbolic logic, indefinite metric, measure, indefinite measure, trace.

THEOREM. *Let $\mathcal{A}$ be a W^*J–factor different of I_2. Then for any finite indefinite measure $\mu\colon \Pi_f \to \mathbb{R}$ there exist a J–selfadjoint trace-class operator T and a semiconstant measure μ_* such that $\mu(p) = \tau(Tp) + \mu_*(p)$, $\forall p \in \Pi_f$. If the projections P^+ and P^- are infinite relative to $\mathcal{A}$, then $\mu^* \equiv 0$.*

Proof. One can suppose that there exists a partial isometry $v \in \mathcal{A}$ such that $vp^+v^* \le p^-$. Let $g \in \Pi_f$ be an orthogonal projection and let $g_+ > 0$, $g_- > 0$. From the theorem of the paper [1] it holds that there exist a unique J–selfadjoint trace-class operator T_g $(T_g = gT_gg)$ and a number c_g such that $\mu(e) = \tau(T_ge) + c_g\tau(e_+)$, $\forall e \le g$. If the projections g_+ and g_- are both infinite relative to $\mathcal{A}$, then $c_g = 0$ $(0 \cdot \infty = 0)$.

Hence, if $p \in \Pi_f$ is an orthogonal projection and $g \le p$, then $gT_pg = T_g$ and $c_p = c_g$. Put $c = c_p$.

Now we prove that $c = 0$ if the projections P^+ and P^- are both infinite relative to $\mathcal{A}$. Let $\{e_\iota\}_{\iota \in I}$ be an infinite set of pairwise orthogonal finite projections from $\mathcal{A}$, and $\sum\limits_\iota e_\iota = P^+$. Put $\varphi_\iota \equiv ve_\iota v^*$. The operators

$$q\big(\tfrac{3}{2}\varphi_\iota \pm v^*\varphi_\iota\big) \equiv \tfrac{1}{2}(3\varphi_\iota \pm \sqrt{3}v^*\,\varphi_\iota \mp \sqrt{3}\varphi_\iota v - v^*\varphi_\iota v)$$

are projections from Π^-. Let $T_\iota \equiv T_{\varphi_\iota + e_\iota}$. The operator T_ι is J–selfadjoint. Hence,

$$\begin{aligned}\mu\Big(q\big(\tfrac{3}{2}\varphi_\iota \pm v^*\varphi_\iota\big)\Big) &= \frac{1}{2}\big(3\tau(T_\iota\varphi_\iota) \pm \sqrt{3}\tau\big(T_\iota(v^*\varphi_\iota - \varphi_\iota v)\big) - \tau(T_\iota e_\iota)\big) = \\ &= \frac{1}{2}\big(3\mu(\varphi_\iota) \pm 2\sqrt{3}\,\mathrm{Re}\,\tau(T_\iota Jv^*\varphi_\iota) - (\mu(e_\iota) - c\tau(e_\iota))\big)\ (\le 0).\end{aligned}$$

Let

$$X \equiv \big\{\iota \in I\colon \big(\mu(e_\iota) - c\tau(e_\iota)\big)\,\mathrm{Re}\,\tau(T_\iota Jv^*\varphi_\iota) \le 0\big\}.$$

The projections from $\big\{q(\tfrac{3}{2}\varphi_\iota\, v^*\varphi_\iota)\big\}_{\iota \in X} \cup \big\{q(\tfrac{3}{2}\varphi_\iota\, v^*\varphi_\iota)\big\}_{\iota \in I\setminus X}$ are by the construction pairwise orthogonal. Hence there exists the projection

$$q \equiv \sum_{\iota \in X} q\Big(\tfrac{3}{2}\varphi_\iota, v^*\varphi_\iota\Big) + \sum_{\iota \in I\setminus X} q\Big(\tfrac{3}{2}\varphi_\iota, -v^*\varphi_\iota\Big) \in \Pi^-.$$

This implies

$$M \equiv \frac{1}{2}\Big(\sum_{\iota \in X} \big|\mu(q(\tfrac{3}{2}\varphi_\iota, v^*\varphi_\iota))\big| + \sum_{\iota \in I\setminus X} \big|\mu(q(\tfrac{3}{2}\varphi_\iota, -v^*\varphi_\iota))\big|\Big) < +\infty.$$

From this it follows

$$0 \le \frac{1}{2}\sum_{\iota\in I}|\mu(e_\iota)-c\tau(e_\iota)| \le \frac{1}{2}\Big(\sum_{\iota\in X}|2\sqrt{3}\operatorname{Re}\tau(T_\iota Jv^*\varphi_\iota)-(\mu(e_\iota)-c\tau(e_\iota))|+$$

$$+\sum_{\iota\in I\setminus X}|-2\sqrt{3}\operatorname{Re}\tau(T_\iota Jv^*\varphi_\iota)-(\mu(e_\iota)-c\tau(e_\iota))|\Big) \le M+\frac{3}{2}\sum_{\iota}|\tau(T_\iota\varphi_\iota)| =$$

$$= M-\frac{3}{2}\mu\Big(\sum_\iota\varphi_\iota\Big) < +\infty\,.$$

In addition, $0 \le \sum_\iota \mu(e_\iota) < +\infty$. Hence

$$|c|\sum\tau(e_\iota) = |c|\tau(p^+) = |c|\cdot +\infty < +\infty\,.$$

This implies $c=0$.

Now we show there exists a J–selfadjoint trace-class operator T such that $gTg = T_g$ for any orthogonal projection $g\in\Pi_f$. Any measure μ can be represented as the sum of a Hermitian component $\mu_h(e)\equiv\frac{1}{2}(\mu(e)+\mu(e^*))$, $\forall e$ and a skew-Hermitian component $\mu_s(e)\equiv\frac{1}{2}(\mu(e)-\mu(e^*))$, $\forall e$. Hence $\mu_h(e)=\tau\left(\frac{1}{2}(T_g+T_g^*)e\right)+c\tau(e_+)$ and $\mu_s(e)=\tau\left(\frac{1}{2}(T_g-T_g^*)e\right)$, $\forall e\le g$. The operator $T_{hg}=\frac{1}{2}(T_g+T_g^*)$ is selfadjoint and J–selfadjoint and the operator $T_{sg}=\frac{1}{2}(T_g-T_g^*)$ is skew-adjoint and J–skew-adjoint (such as $T_g=JT_g^*J$). This implies

$$T_{hg}=P^+T_{hg}P^+ + P^-T_{hg}P^- \quad \text{and}$$

$$T_{sg}=P^+T_{sg}P^- + P^-T_{sg}P^+ = U|P^-T_{sg}P^+| - |P^-T_{sg}P^+|U^+,$$

where $P^-T_{sg}P^+ = U\,|P^-T_{sg}P^+|$ polar decomposition of the operator $P^-T_{sg}P^+$.

By the theorem of the paper [3], there exist selfadjoint trace-class operators A^+, A^- such that $A^\pm=P^\pm A^\pm P^\pm$ and

$$\mu_h(e)=\tau(A^+e)+c\tau(e),\quad \forall e\le P^+,$$

$$\mu_h(e)=\tau(A^-e),\quad \forall e\le P^-.$$

Hence, there exist sequences of finite orthogonal projections $\{e_n^+\}$ and $\{e_n^-\}$, $e_n^+\le P^+$, $e_n^-\le P^-$, $\forall n$, $e_n^\pm\downarrow 0$ such that the operators $(e_n^+)^\perp A^+(e_n^+)^\perp$ and $(e_n^-)^\perp A^-(e_n^-)^\perp$ are bounded. Then by the Lemma 1 [1], the following inequality

$$|\tau((e_n^++e_n^-)^\perp T_{sg}(e_n^++e_n^-)^\perp(r-r^*))| = 2|\,\mu_s(r)| \le$$

$$\le 48\|r\|\sup\{\mu(e^+)-\mu(e^-)\colon e^+\le P^+,\, e^-\le P^-\},$$

where $r \in \Pi_f$, $r \le g \le (e_n^+ + e_n^-)^\perp$, holds. From this it follows that there exists a constant t such that $\|T_{sg}\| \le t$ for any projection $g \le (e_n^+ + e_n^-)^\perp$. Hence there exist a skew-adjoint and a J–skew-adjoint bounded operator T_s^n for which $(e_n^+ + e_n^-)^\perp T_s^n (e_n^+ + e_n^-)^\perp = T_s^n$ and $gT_s^n g = T_{sg}$, $\forall g$, $g \le (e_n^+ + e_n^-)^\perp$, $T_s^n = (e_n^+ + e_n^-)^\perp T_s^m (e_n^+ + e_n^-)^\perp$, where $m > n$. Hence, $\lim_{n\to\infty} T_s^n = T_s$ exists in τ-topology. Let now $T \equiv T_s + A^+ + A^-$. The operator T is that in question. □

REFERENCES

[1] MATVEJCHUK, M. S.: *A description of indefinite measures in W^*J–factors*, Dokl. Akad. Nauk SSSR **38** (1991), 558–561. (In Russian)

[2] MATVEJCHUK, M. S.: *A description of indefinite measures in W^*J–factors*, Engl. transl., Soviet Math. Dokl. **44** 1 (1992), 161–165.

[3] MATVEJCHUK, M. S.: *Description of finite measures on semifinite algebras*, Funkt. Anal. Priloz. **15** (1981), 41–53. (In Russian)

[4] MATVEJCHUK, M. S.: *Description of finite measures on semifinite algebras*, Functional Anal. Appl. **15** 3 (1981), 187–197.

Received February 1, 1993

Kazan University
Math.-Mech. Faculty
Lenin str. 18
SU 420008 Kazan
RUSSIA