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A NOTE ON THE $\oplus$–MEASURE BASED INTEGRALS

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ABSTRACT. The structure of the pseudo-addition $\oplus$ introduced by Marinová in [5] is studied. Some properties of integrals with respect to $\oplus$-measures are shown.

1. Introduction

In 1983, I. Marinová introduced an integral with respect to a $\oplus$-measure (published in Math. Slovaca 36 (1986), No.1, 15–22).
The concept of $\oplus$-measures was suggested by Z. Riečanová in [9]. The notion of $\oplus$-measures arose as a common generalization of σ-additive and σ-maxitive measures.

One of Marinová's sources was Shilkret's paper [11]. The set functions considered in this paper had the "maxitivity" property $m(\cup E_n) = \sup m(E_n)$ instead of the usual additivity. The integral of Marinová represents one of the first attempts of defining an integral that would be an extension of the Lebesgue integral and would include integration with respect to some non-additive set functions.

There are some other integrals based on non-additive set functions. Some of them are in [3], [7], [12], [13]. E.g., Weber [13] proposed the extended Lebesgue integral with respect to a $\perp$-decomposable measure in the case that $\perp$ is a continuous Archimedean t-conorm. Note that a $\perp$-decomposable measure in [13] is a set function $m\colon \mathcal{S} \to [0,1]$, $m(\emptyset) = 0$, $m(\Omega) = 1$, and $m(A \,\dot\cup\, B) = m(A) \perp m(B)$, where $(\Omega, \mathcal{S})$ is a measurable space and $\dot\cup$ denotes the union of disjoint sets $A, B \in \mathcal{S}$. Sugeno and Murofushi presented in [12] the extended Lebesgue integral with respect to a pseudo-additive measure replacing the ordinary addition $+$ by a pseudo-addition $\hat{+}$ and the ordinary multiplication $\cdot$ by a multiplication $\hat{\cdot}$ corresponding to the operation $\hat{+}$. Note that the case

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$(\hat{+}, \hat{\cdot}) = (\vee, \cdot)$ leads to the Shilkret's integral and the case $(+, \cdot)$ leads to the Lebesgue integral. $\oplus$-measures considered in [5] are based on a special type of a pseudo-addition $\oplus$ (on the interval $[0,\infty]$). The aim of the presented paper is to study the structure of the operation $\oplus$ used by Marinová and to specify all such operations. Moreover, in Section 3 some properties of $\oplus$-measure based integrals (which follow just from the structure of the operation $\oplus$) will be shown.

2. Structure of the operation $\oplus$

Let us consider a binary operation $\oplus$ on $[0,\infty]$ with the following properties:

(A1) $a \oplus b = b \oplus a$,
(A2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
(A3) $k \cdot (a \oplus b) = (k \cdot a) \oplus (k \cdot b)$,
(A4) $a \oplus 0 = a$, $a \oplus \infty = \infty$,
(A5) $a \leq b \Rightarrow a \oplus c \leq b \oplus c$,
(A6) $(a+b) \oplus (c+d) \leq (a \oplus c) + (b \oplus d)$,
(A7) $a_n \to a$ and $b_n \to b \Rightarrow a_n \oplus b_n \to a \oplus b$

for each $a, b, c, d, a_n, b_n \in [0,\infty]$, $(n = 1, 2, \dots)$ and each $k > 0$.

Note that "$\leq$" means the usual order of real numbers and the symbol "$\cdot$" in (A3) is used for the ordinary multiplication. We will omit it if there are no doubts.

It is evident that the ordinary addition $+$ and the maximum $\vee$ of two numbers satisfy the properties (A1)–(A7). But as we can see later these operations are marginal in certain sense.

The binary operation $\oplus$ on $[0,\infty]$ with the properties (A1)–(A7) is a special type of a continuous pseudo-addition $\hat{+}$ on $[0,\infty]$, it means a binary operation which is commutative, associative, non-decreasing, continuous, with 0 as a neutral element.

It is known that each continuous pseudo-addition $\hat{+}$ can be represented by a family of one-place functions as follows:

Lemma 1. *Let $\hat{+}$ be a continuous pseudo-addition on $[0,\infty]$. Then there exist a system $\{(a_i, b_i);\ i \in I\}$ of open disjoint intervals in $[0,\infty]$ and a system $\{g_i;\ i \in I\}$ of continuous strictly increasing functions $g_i : [a_i, b_i] \to [0,\infty]$, $g_i(a_i) = 0$, such that*

$$x \hat{+} y = \begin{cases} \tilde{g}_i^{-1}(g_i(x) + g_i(y)), & \text{for } x, y \in \langle a_i, b_i \rangle^2, \\ \max(x, y), & \text{otherwise}, \end{cases}$$

where $\tilde{g}_i^{-1}$ *is the pseudo-inverse of* g_i *defined by*

$$\tilde{g}_i^{-1}(x) = g_i^{-1}\big(\min(x, g_i(b_i))\big) .$$

More details can be found in [12]. Note that the functions g_i, $i \in I$, are called generators.

The system $\{\langle (a_i, b_i),\, g_i\rangle\,;\ i \in I\}$ from Lemma 1 is briefly called a representation of the operation $\hat{+}$.

For example, the ordinary addition $+$ has the representation $\{\langle (0,\infty), \mathrm{id}\rangle\}$. It means that this operation is generated on the whole interval $[0,\infty]$ by the function $g(x) = x$. The representation of the maximum $\vee$ is $\emptyset$.

PROPOSITION 1. *Let* $\oplus$ *be a binary operation on* $[0,\infty]$ *with the properties (A1)–(A7) which is different from* $\vee$, *i.e., with a non-empty representation. Then the representation of the operation* $\oplus$ *has the form*

$$\{\langle (0,\infty),\, g\rangle\}, \quad g(\infty) = \infty ,$$

i.e.,

$$x \oplus y = g^{-1}\big(g(x) + g(y)\big) \qquad \text{for each} \quad x, y \in [0,\infty] .$$

Proof. Let $\{\langle (\alpha_i, \beta_i), g_i\rangle\,,\ i \in I\} \neq \emptyset$ be a representation of the operation $\oplus$.

Let $j \in I$ be such that $\beta_j \neq \infty$. Let us take such $x, y \in (\alpha_j, \beta_j)$ that $x < y$ and $g_j(x) + g_j(y) < g_j(\beta_j)$. Then $x \oplus y = g_j^{-1}\big(g_j(x) + g(y)\big)$.

Let k be such a positive constant that $kx \in (\alpha_j,\, \beta_j)$ and $ky > \beta_j$. From this assumption it follows $kx \oplus ky = \max(kx, ky) = ky$. By the property (A3) $kx \oplus ky = k(x \oplus y)$, so we have $x \oplus y = y$. It means that

$$g_j^{-1}\big(g_j(x) + g_j(y)\big) = y$$

what implies $g_j(x) = 0$. This is impossible, since g_j is a strictly increasing function. So we have $\beta_j = \infty$, $I = \{j\}$ and $(\alpha_j, \beta_j) = (\alpha, \infty)$. Similarly we get $\alpha = 0$. It means that the representation of $\oplus$ is of the form $\{\langle (0,\infty),\, g\rangle\}$.

Now let us consider $g(\infty) = M < \infty$. Let $\omega \in (0,\infty)$ be such that $g(\omega) = M/2$ and let $0 < x < y < \omega$.

Since $g(x) + g(y) < M$, $x \oplus y = g^{-1}\big(g(x) + g(y)\big)$ is a real number. Let k be a positive constant for which $\omega < kx < ky$. Then since $g(kx) + g(ky) > M$, it holds

$$kx \oplus ky = g^{-1}(M) = \infty .$$

This is impossible, since by (A3) and the previous result $kx \oplus ky$ is a real number.

Now the proof is complete. □

By the previous statement a binary operation on $[0,\infty]$ with the properties (A1)–(A7) is either $\vee$ or a strict continuous t-conorm on $[0,\infty]$.

PROPOSITION 2. *Let $\oplus$ be a binary operation on $[0,\infty]$ with the properties (A1)–(A7) and let $\{\langle(0,\infty),\, g\rangle\}$ be its representation. Then*

$$g(x) = ax^r \qquad \text{for some} \quad r \geq 1 \qquad \text{and} \quad a > 0\,.$$

Proof. By the property (A3) the equation $k(x \oplus y) = kx \oplus ky$ holds for each positive constant k and each $x, y \in [0,\infty]$. Using Proposition 1 we get

$$x \oplus y = \frac{1}{k} g^{-1}\big(g(kx) + g(ky)\big)\,.$$

If we denote

$$h_k \colon [0,\infty] \longrightarrow [0,\infty], \quad h_k(x) = g(kx)$$

for each $k > 0$, then $h_k^{-1}(x) = \frac{1}{k} g^{-1}(x)$ and the right side of the previous equation can be written in the form:

$$\frac{1}{k}\, g^{-1}\big(g(kx) + g(ky)\big) = h_k^{-1}\big(h_k(x) + h_k(y)\big)\,,$$

so we have

$$x \oplus y = h_k^{-1}\big(h_k(x) + h_k(y)\big)\,.$$

This means that h_k is a generator of the operation $\oplus$. But a generator of the operation $\oplus$ is unique up to a positive multiplicative constant factor [2]. It means for each $k > 0$ there exists a positive constant c_k:

$$h_k(x) = c_k\, g(x), \quad x \in [0,\infty]\,.$$

Since h_k is defined above by $h_k(x) = g(kx)$, we have shown that

$$g(kx) = c_k\, g(x), \qquad \text{for} \quad k > 0,\ x \in [0,\infty]\,.$$

As c_k is a constant depending only on k, we can write $c_k = c(k)$, $k > 0$. The continuity of the function g implies the continuity of the function c. Moreover, it holds : $g(k) = c(k)g(1)$. This implies

$$g(xy) = c(x)\, g(y) = c(x)\, c(y) g(1)$$

and

$$g(xy) = c(xy)\, g(1)$$

for positive x, y. The consequence of these relations is:

$$c(xy) = c(x)\,c(y) \qquad \text{for} \quad x, y > 0\,.$$

Since c is a non-negative, continuous function defined on $(0, \infty)$, satisfying a functional equation $c(xy) = c(x)\,c(y)$ for all x, y positive, there exists [1] a real number r such that

$$c(x) = x^r, \quad x \in (0, \infty)\,.$$

It means

$$g(x) = ax^r, \quad a = g(1)\,.$$

Since a generator g is a strictly increasing function, $r > 0$. Any t-conorm generated by a generator $g(x) = ax^r$, $a > 0$, $r > 0$, satisfies the properties (A1)–(A5) and (A7). Recall the property (A6):

$$(a+b) \oplus (c+d) \le (a \oplus c) + (b \oplus d) \qquad \text{for all} \quad a, b, c, d \in [0, \infty], \text{ i.e.,}$$
$$g^{-1}\big(g(a+b) + g(c+d)\big) \le g^{-1}\big(g(a) + g(c)\big) + g^{-1}\big(g(b) + g(d)\big)\,.$$

Putting $a = d = 0$ and $b = c = x$ we get:

$$g^{-1}\big(2g(x)\big) \le 2x \qquad \text{or} \quad 2g(x) \le g(2x) \qquad \text{for all} \quad x > 0\,,$$

i.e.,

$$2ax^r \le a(2x)^r \qquad \text{and} \quad 2 \le 2^r\,.$$

The last inequality holds if and only if $r \ge 1$.

So we have shown that if g is a generator of the operation $\oplus$, then g is given by $g(x) = ax^r$, for some $a > 0$ and $r \ge 1$. □

Moreover the following assertion is true.

PROPOSITION 3. *For each $r \ge 1$ the binary operation $\oplus_r$ on $[0, \infty]$ defined by*

$$x \oplus_r y = (x^r + y^r)^{1/r}$$

has the properties (A1)–(A7).

P r o o f. Let $g(x) = x^r$, $r \ge 1$. It is obvious that the binary operation $\oplus_r$ has the properties (A1)–(A5) and (A7).

We still have to show that the property (A6): $(a+b) \oplus_r (c+d) \le (a \oplus_r c) + (b \oplus_r d)$ is valid for each $a, b, c, d \in [0, \infty]$.

Let us denote $a = x$, $c = y$, $a+b = u$, $c+d = v$. Then (A6) can be written in the form:

$$(u^r + v^r)^{1/r} \le (x^r + y^r)^{1/r} + \big[(u-x)^r + (v-y)^r\big]^{1/r}\,.$$

Let $h(x,y) = (x^r + y^r)^{1/r} + \left[(u-x)^r + (v-y)^r\right]^{1/r}$. By using partial derivatives of the function $h(x,y)$ we get a condition for the minimum of h: $uy = vx$ and then

$$h_{\min} = (u^r + v^r)^{1/r} .$$

It means $(u^r + v^r)^{1/r} \le h(x,y)$, what is in fact (A6). □

Remark 1. Let r be a fixed number, $r \ge 1$. Then each function $g(x) = ax^r$, $a > 0$, is a generator of the operation $\oplus_r$ defined above. The generator $g(x) = x^r$ (i.e., for $a = g(1) = 1$) is called a normed generator.

THEOREM 1. *A binary operation $\oplus$ on $[0,\infty]$ has the properties (A1)–(A7) if and only if $\oplus = \vee$ or $\oplus = \oplus_r$ for some $r \ge 1$.*

Proof. The proof of this theorem follows immediately from Propositions 1, 2 and 3.

Let us only show that if $\oplus \ne \vee$ then according to Proposition 2 the generator g of $\oplus$ is given by $g(x) = a \cdot x^r$ for some $r \ge 1$ and $a > 0$. Since $g^{-1}(u) = \left(\frac{u}{a}\right)^{1/r}$ we get (Proposition 1):

$$x \oplus y = g^{-1}\big(g(x) + g(y)\big) = \left(\frac{ax^r + ay^r}{a}\right)^{1/r} = (x^r + y^r)^{1/r} ,$$

i.e., $\oplus = \oplus_r$ for some $r \ge 1$. □

3. Consequences for integrals with respect to $\oplus$-measures

First let us recall the definition of a $\oplus$-measure and the way of defining the integral with respect to a $\oplus$-measure from [5].

Let $(X, \mathcal{S})$ be a measurable space, i.e., let X be an arbitrary non-empty set and let $\mathcal{S}$ be a σ-algebra of its subsets.

DEFINITION 1. A set function $m\colon \mathcal{S} \to [0,\infty]$ is said to be a $\oplus$-measure if it has the following two properties:

(i) $m(\emptyset) = 0$,

(ii) if $\{A_n\}_{n\in\mathrm{N}} \subset \mathcal{S}$, $A_i \cap A_j = \emptyset$ for $i \ne j$, then

$$m\Big(\bigcup_{n\in\mathrm{N}} A_n\Big) = \sup_{n\in\mathrm{N}}\big\{m(a_1) \oplus m(A_2) \oplus \cdots \oplus (A_n)\big\} ,$$

where $\oplus$ is a binary operation on $[0,\infty]$ with the properties (A1)–(A7).

Definition 2.

(i) If s is a simple non-negative measurable function defined on X, $s = \sum_{i=1}^{n} a_i 1_{A_i}$, then the integral of s with respect to m is given by

$$\int_X s \bullet m = a_1 m(A_1) \oplus a_2 m(A_2) \oplus \cdots \oplus a_n m(A_n) \tag{1}$$

or briefly

$$\int_X s \bullet m = \bigoplus_{i=1}^{n} a_i m(A_i) .$$

(ii) If f is a non-negative measurable function defined on X, then

$$\int_X f \bullet m = \sup\left\{ \int_X s \bullet m;\ s\colon \text{ simple, non-neg., measurable, } s \le f \right\} . \tag{2}$$

Due to explaining the structure of the operation $\oplus$, we can give some more results about the integrals with respect to $\oplus$- measures.

First we state without the proof the following simple lemma.

Lemma 2. *Let m be a $\oplus$-measure on $(X, \mathcal{S})$ and let g be a generator of the operation $\oplus$. Then a set function*

$$g \circ m \colon \mathcal{S} \to [0, \infty], \quad g \circ m(A) = g\big(m(A)\big) ,$$

is a σ-additive measure.

Proposition 4. *Let s be a simple non-negative measurable function on X, let m be a $\oplus$-measure on $(X, \mathcal{S})$, $\oplus \ne \vee$ and let g be a normed generator of the operation $\oplus$. Then*

$$\int_X s \bullet m = g^{-1}\left(\int_X (g \circ s)\, d(g \circ m) \right) , \tag{3}$$

where the right-hand side integral is the Lebesgue one.

P r o o f. Let $s = \sum_{i=1}^{n} a_i 1_{A_i}$, $a_i > 0$, $A_i \in \mathcal{S}$. Then by (1)

$$I = \int_X s \bullet m = \bigoplus_{i=1}^{n} a_i m(A_i) .$$

If we use the equation $x \oplus y = g^{-1}\big(g(x) + g(y)\big)$ we get

$$I = g^{-1}\left(\sum_{i=1}^{n} g\big(a_i m(A_i)\big)\right).$$

Since the normed generator of the operation $\oplus$ has the form $g(x) = x^r$ for some $r \geq 1$ we have

$$I = g^{-1}\left(\sum_{i=1}^{n} a_i^r\, m(A_i)^r\right).$$

The previous formula can be written in the form

$$I = g^{-1}\left(\sum_{i=1}^{n} g(a_i)\, g\big(m(A_i)\big)\right) = g^{-1}\left(\sum_{i=1}^{n} g(a_i)\, g_{\circ}m(A_i)\right) =$$
$$= g^{-1}\left(\int_X (g_{\circ}s)\, d(g_{\circ}m)\right),$$

where the integral in the last formula is the Lebesgue integral of a simple function $g_{\circ}s$ with respect to a σ-additive measure $g_{\circ}m$ (Lemma 2). □

THEOREM 2. *Let $(X, \mathcal{S})$ be a measurable space, let m be a $\oplus$-measure on $(X, \mathcal{S})$, $\oplus \neq \vee$, and let g be a normed generator of the operation $\oplus$. Then the integral of a non-negative measurable function f on X is given by:*

$$\int_X f \bullet m = g^{-1}\left(\int_x (g \circ f)\, d(g \circ m)\right). \tag{4}$$

The proof of this theorem is based on using Proposition 4 and Definition 2 (ii). We omit the details. □

EXAMPLE 1. Let $X = [0, \infty]$. Consider the $\oplus_2$-measure m defined by $m(A) = \lambda(A)^{1/2}$, where λ is the Lebesgue measure.

(i) Let $s = 2.1_{[0,1]} + 3.1_{(1,3)}$. Then by the definition of an integral of a simple function we have

$$\int_X s \bullet m = 2 \cdot m(\langle 0,1\rangle) \oplus_2 3 \cdot m((1,3\rangle) = 2 \oplus_2 3 \cdot \sqrt{2} = \sqrt{4 + 18} = \sqrt{22}.$$

Using (3) from Proposition 4 we get

$$\int_X s \bullet m = \left(\int_X s^2 d\lambda\right)^{1/2} = \sqrt{4.1 + 9.2} = \sqrt{22}.$$

(ii) If $f(x) = 1/x$ for $x \in [1, \infty)$ and $f(x) = 0$ otherwise, then

$$\int_X f \bullet m = \left[\int_1^\infty (1/x^2) d\lambda\right]^{1/2} = 1 .$$

Remark 2. (i) If we use an arbitrary generator g of the operation $\oplus$, i.e., $g(x) = ax^r$, $a > 0$, $r \geq 1$, then the formula (4) must be changed into the following form:

$$\int_X f \bullet m = g^{-1}\left[(1/a) \int_X (g \circ f)\, d(g \circ m)\right].$$

(ii) We have shown that the integral of Marinová which is based on a binary operation $\oplus$ with the properties (A1)–(A7), on the ordinary multiplication and $\oplus$-measure is a special case of the Pap integral [7] on $[0, \infty]$.

Theorem 2 implies directly some results of Marinová, e.g. the monotone convergence theorem or the Lebesgue-like convergence theorem, proofs of which are now due to Theorem 2 very simple. Let us show it at least for the first of the mentioned theorems.

PROPOSITION 5. *Let $(X, \mathcal{S})$ be a measurable space, m be a $\oplus$-measure on $(X, \mathcal{S})$, $\oplus \neq \vee$. Let $\{f_n\}_{n \in \mathrm{N}}$ be a sequence of non-negative integrable functions such that $f_n \nearrow f$ and $\lim\limits_{n\to\infty} \int_X f_n \bullet m < \infty$. Then the function f is integrable and*

$$\int_X f \bullet m = \lim_{n\to\infty} \int_X f_n \bullet m .$$

Proof. Since $\oplus \neq \vee$ there exists $r \geq 1$ such that $\oplus = \oplus_r$. Let g be a normed generator of the operation $\oplus_r$. Then, using Theorem 2, properties of the generator g and the monotone convergence theorem for the Lebesgue integral, we get:

$$\begin{aligned}
\lim_{n\to\infty} \int_X f_n \bullet m &= \lim_{n\to\infty} g^{-1}\left(\int_X (g \circ f_n)\, d(g \circ m)\right) = \\
&= g^{-1}\left(\lim_{n\to\infty} \int_X (g \circ f_n)\, d(g \circ m)\right) = \\
&= g^{-1}\left(\int_X (g \circ f)\, d(g \circ m)\right) = \int_X f \bullet m .
\end{aligned}$$

□

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