Tatra Mountains Math. Publ. **3** (1993), 251–264

TATRA MOUNTAINS
Mathematical Publications

FOURIER–WALSH SERIES OF VECTOR–VALUED MEASURES AND FUNCTIONS

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ABSTRACT. Let $\{a_n\}$ be a sequence of elements of a locally convex vector space X. The paper settles the problem of the existence of such a vector-valued function and a measure, that $\{a_n\}$ are their Fourier–Walsh coefficients on the unit interval.

0. Introduction

Recently the Walsh functions and their applications have drawn much attention from both mathematicians and engineers alike. They form a complete orthonormal system and since there are real-time algorithms for obtaining the coefficient sequences of the Walsh series of any square integrable function f and for recovering f from these sequences, their applications seems to be endless.

In 1949, Fine made the fundamental observation that the Walsh functions can be viewed as characters of a dyadic group. A similar dichotomy prevails for the classical Fourier analysis. One can investigate a trigonometric series on the interval $\langle O, 2\pi)$ or an exponential series on the circle group T. This identification allows us to translate results from one system to other one.

Particularly, let $\{a_n\}_{n=0}^{\infty}$ be elements of a vector space X and let I denote the interval $\langle 0, 1)$ (mod 1). The purpose of this paper is to answer to the following questions:

1. Does there exist a vector-valued measure on I such that a_n's are the Fourier-Walsh-Stieltjes coefficients of this measure.
2. Does there exist a vector-valued function on I such that a_n's are the Fourier-Walsh coefficients of this function.

In the first, we consider the above problems for continuous orthonormal systems of characters on the dyadic group in terms of the Cesaro means. The analogous problem for the Walsh system needs a reformulation since the Walsh functions are not continuos on I. We shall proceed similarly as Fine in [1], who solved a version of this problem for the scalar case.

AMS Subject Classification (1991): 42C10, 43A25, 43A40.
Key words: Fourier–Walsh–Stieltjes coefficients, dyadic group, Walsh system.

1. Notation and definitions

We shall use the notation of [2]. Let $\mathbb{N}$ denote the set of all non-negative integers.

The *dyadic group* G consists of all sequences $\overline{x} = (x_1, x_2, \ldots)$, $x_i \in \{0,1\}$ where addition is defined coordinatewise mod 2. We consider G with the product topology. Hence, G is a totally disconnected compact topological group.

The terms "Walsh functions" refer to one of three orthonormal systems: the Walsh–Paley system, the original Walsh system or the Walsh-Kaczmarz system. This systems contain the same functions and differ only in ordering. Each is a complete orthonormal system on $\langle 0,1)$ and contains the Rademacher system.

The *Walsh–Paley system* $w := (w_n,\ n \in \mathbb{N})$ is defined as the product of Rademacher functions in the following way. If $n \in \mathbb{N}$ has binary coefficients $(n_k, k \in \mathbb{N})$, then

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k},$$

where

$$r(x) = \begin{cases} 1, & x \in \langle 0, \frac{1}{2}), \\ -1, & x \in \langle \frac{1}{2}, 1), \end{cases}$$

$$\begin{aligned} r(x+k) &= r(x), \quad x \in \langle 0,1),\ k \in \mathbb{N}, \\ r_k(x) &= r(2^k x), \quad x \in \mathbb{R},\ k \in \mathbb{N}. \end{aligned}$$

We describe the characters of G. For $n \in \mathbb{N}$ and $\overline{x} = (x_1, x_2, \ldots) \in G$, the collection $(\xi_n, n \in \mathbb{N})$

$$\xi_n(x) := (-1)^{x_n}$$

generates all characters of the dyadic group G in the same way that the Rademacher functions generate the Walsh system.

PROPOSITION 1. *For each $n \in \mathbb{N}$ with binary coefficients $(n_k, k \in \mathbb{N})$, let*

$$\psi_n := \prod_{k=0}^{\infty} \xi_k^{n_k},$$

then ψ_n is a character on G, and, conversely, every character is of this form (see [2, Theorem 1]).

Define Fine's map $\rho\colon \langle 0,1) \to G$ by

$$\rho(x) := \overline{x} = (x_1, x_2, \ldots),$$

where x have the dyadic expansion

$$x = \sum_{k=1}^{\infty} x_k 2^{-k}, \quad x_k \in \{0,1\}.$$

For each $x \in \langle 0,1) \setminus Q$, where $Q \subset I$ ($\overline{Q} \subset G$) presents the set of all dyadic rationals, there is only one expression of this form. The map ρ is well defined by choosing the finite expansion in case of doubt. When $x \in Q$, there are two expressions of this form, one which terminates in 0's and one which terminates in 1's. By the dyadic expansion of $x \in Q$ we mean the one which terminates in 0's.

We make the inverse $\lambda\colon G \to \langle 0,1)$ via

$$\lambda(\overline{x}) = \sum_{i=1}^{\infty} x_i 2^{-i} .$$

The set of all $\overline{x}$ in $\overline{Q}$, such that $\lambda(\overline{x})$ terminates in 1's, we denote by $\overline{Q}'$.

If f is real-valued on I, then there is a corresponding function $\overline{f}$ on G, given by

$$\overline{f}(\overline{x}) = \begin{cases} f\big(\lambda(\overline{x})\big), & \overline{x} \in G \setminus \overline{Q}', \\ \limsup\limits_{\overline{y} \to \overline{x}} \overline{f}(\overline{y}), & \overline{x} \in \overline{Q}', \end{cases} \tag{1}$$

where the approach is over those $\overline{y}$ corresponding to dyadic irrationals. We indicate that (1) holds by writing $\overline{f} \sim f$.

On the contrary, if we have $\overline{f}\colon G \to R$, then for $f\colon I \to R$

$$f(x) = \overline{f}\big(\rho(x)\big), \quad x \in \langle 0,1) .$$

If f is continuous so is $\overline{f}$, but not conversely. That is $w_k \sim \psi_k$ and

$$w_k = \psi_k \circ \rho, \quad x \in \langle 0,1),$$
$$\psi_k = w_k \circ \lambda, \quad \overline{x} \in G \setminus \overline{Q}' .$$

The characters ψ_k are continuous but the corresponding Walsh functions w_k are in $C_W(I)$, i.e., continuous only at every dyadic irrational, continuous from the right on I and have a finite limit from the left on I, all this in the usual topology. For integrable functions f and $\overline{f}$, $f \in L_p(I)$ if and only if $\overline{f} \in L_p(G)$.

By a *measure* on G (on I) we shall mean a real finite signed measure μ (m on I) on the Borel sets in G (in I). Every measure on G can be decomposed uniquely into a usual measure, vanishing on all subsets of $\overline{Q}'$ and an unusual measure, vanishing on all Borel subsets of $G \setminus \overline{Q}'$ (see [1]). There is a one-to-one correspondence, denoted by $\mu \sim m$, between the usual measure on G and the measure on I given by

$$\mu(A) = \begin{cases} m\lambda(A), & A \subset G \setminus \overline{Q}', \\ 0, & A \subset \overline{Q}', \end{cases}$$

or by

$$m(B) = \mu\rho(B), \quad B \subset I .$$

If μ is a measure on G and

$$a_k = \int_G \psi_k \, \mathrm{d}\mu ,$$

then $\overline{S} = \sum a_k \psi_k$ is called the *Fourier-Walsh-Stieltjes series of the measure* μ and we write $\overline{S} = \overline{S}(\mathrm{d}\mu)$.

If m is a measure on I and

$$a_k = \int_I w_k \, \mathrm{d}m ,$$

then $S = \sum a_k w_k$ is called the *Fourier-Walsh-Stieltjes series of the measure* m and we write $S = S(\mathrm{d}m)$. In both cases the measure is determined uniquely on Borel sets by the sequence a_k.

If a series $\overline{S}$ and S have the same coefficients, we write $\overline{S} \sim S$, or $S \sim \overline{S}$.

2. Fourier–Walsh series of X-valued measures

Let $\mathcal{B}(G)$ $(\mathcal{B}(I))$ be the σ-algebra of all Borel sets in G (in I).

THEOREM 1.

(i) *Let* $f\colon I \to X$ *be Pettis-integrable, let* $\overline{f}\colon G \to X$ *and* $f \sim \overline{f}$, *then* $\overline{f}$ *is Pettis-integrable and* $\int_G \overline{f} \, \mathrm{d}\overline{x} = \int_I f \, \mathrm{d}x$.

Let $f\colon I \to X$ *be Bochner-integrable,* $\overline{f}\colon G \to X$ *and* $f \sim \overline{f}$ *then* $\overline{f}$ *is Bochner-integrable and* $\int_G \|\overline{f}(\overline{x})\| \, \mathrm{d}\overline{x} = \int_I \|f(x)\| \, \mathrm{d}x$, *where* $\mathrm{d}\overline{x}$ *denotes the normalized Haar measure on* G *and* $\mathrm{d}x$ *denotes the Lebesgue measure on* I.

(ii) *Let* $f\colon I \to R$ *be Bochner or Pettis-integrable,* $m\colon \mathcal{B}(I) \to X$, $\overline{f} \sim f$, $m \sim \mu$ *and* $\int_I f \, \mathrm{d}m$ *exists, then* $\int_G \overline{f} \, \mathrm{d}\mu$ *exists as well, and* $\int_G \overline{f} \, \mathrm{d}\mu = \int_I f \, \mathrm{d}m$.

(iii) $S(\mathrm{d}m_1) = S(\mathrm{d}m_2)$ *implies* $m_1 = m_2$, *and* $\overline{S}(\mathrm{d}\mu_1) = \overline{S}(\mathrm{d}\mu_2)$ *implies* $\mu_1 = \mu_2$ *(on Borel sets).*

(iv) $S = S(\mathrm{d}m)$ *and* $\mu \sim m$ *implies* $S(\mathrm{d}m) \sim \overline{S}(\mathrm{d}\mu)$.

(v) $S(\mathrm{d}m) \sim \overline{S}(\mathrm{d}\mu)$ *implies* $\mu \sim m$.

(vi) $\overline{f}$ *is Pettis-integrable if only if* f *is Pettis-integrable.*

Proof. All the assertions but (i) are easy to prove, because they have a precise analogue in the real case.(See [1, Theorem 1]).

We prove (i). Let $\overline{f}\colon G \to X$ be Pettis-integrable. Hence, there exists a functional $y\colon X' \to R$ in second dual of X such that for all x' in X'

$$y(x') := \int_G \langle \overline{f}(\overline{t}),\ x' \rangle \, \mathrm{d}\overline{t}\,.$$

Since $\overline{f} \sim f$ and the Haar measure vanishes on $\overline{Q}'$, we have by (ii) of this theorem for a real case,

$$\int_G \langle \overline{f}(\overline{t}),\ x' \rangle \, \mathrm{d}\overline{t} = \int_I \langle f(t),\ x' \rangle \, \mathrm{d}t\,.$$

This implies that f is Pettis-integrable. The proof of the converse implication is similar.

Now we prove the other part of Theorem 1 (i). Let $\overline{f}\colon G \to X$ be Bochner-integrable. Hence $\int_G \|\overline{f}(\overline{x})\| \, \mathrm{d}\overline{x}$ exists. Since $\|f(x)\| = \|\overline{f}(\overline{x})\|$ a.e., the proof is finished. □

Now, let X be a quasi-complete, locally convex topological vector space, $C(G)$ be the space of all continuous functions on G with the supreme norm. For each N, let $\Phi_N\colon C(G) \to X$ be a linear map. The set of maps Φ_N is said to be *weakly equi-compact* if there is a weakly compact subset H of X such that

$$\{\Phi_N(\psi);\ \psi \in C(G),\ \|\psi\| \le 1,\ N = 1, 2 \dots\} \subset H.$$

Let $\sigma_N(\overline{t})$ be the $(C,1)$ Cesaro sums of $\overline{S}(d\mu)$. Let ψ_n be the sequence of all characters on G. Then

$$\overline{\sigma}_N(\overline{t}) = \sum_{k=0}^{n-1}\Big(1 - \frac{k}{n}\Big) a_k \psi_k(\overline{t}) = \int_G K_N(\overline{t}, \overline{s}) \, \mathrm{d}\mu(\overline{t})\,,$$

where K_N are the $(C,1)$ kernels, for which

$$K_N(\overline{t}, \overline{s}) = \overline{K_N(\overline{s}, \overline{t})}\,.$$

We shall use the following (see [2]):

Proposition 2. *The sequence of the characters ψ_k is complete, and*

$$\int_G \big|K_N(\overline{t}, \overline{s})\big| \, \mathrm{d}\overline{s} \le 2, \quad t \in G, \quad N = 1, 2, \dots\,.$$

Theorem 2. *The necessary and sufficient condition for a series $\overline{S}$ to be a Fourier-Walsh-Stieltjes series $\overline{S}(d\mu)$ on G, i.e., that there exists a regular measure $\mu\colon \mathcal{B}(G) \to X$ such that the given $a_n \in X$ are the coefficients of μ, is that the set of maps $\Phi_N\colon C(G) \to X$, $N = 1, 2, \dots$, defined by*

$$\Phi_N(\psi) = \int_G \psi(\overline{t}) \overline{\sigma}_N(\overline{t}) \, \mathrm{d}\overline{t}, \quad \psi \in C(G)\,, \tag{2}$$

is weakly equi-compact.

Proof. Suppose that such a measure exists on G. Then for each ψ in $C(G)$

$$\Phi_N(\psi) = \int_G K_N(\bar{t},\bar{s})\,\mu(\mathrm{d}\bar{s})\,\mathrm{d}\bar{t} = \int_G \Big(\int_G K_N(\bar{t},\bar{s})\,\psi(\bar{t})\,\mathrm{d}\bar{t}\Big)\mu(\mathrm{d}\bar{s}).$$

Let $R(\mu) = \{\mu(A);\ A \in \mathcal{B}(G)\}$, (the range of μ) and let W be the closed, absolutely convex hull of $R(\mu)$. Then $R(\mu)$ is relatively weakly compact in X (see [3]) and so, by the Krein theorem, W is weakly compact. Now, for all ψ in $C(G)$ with $\|\psi\| \le 1$, we have, by Proposition 2,

$$\Big|\int_G K_N(\bar{t},\bar{s})\,\psi(\bar{t})\,\mathrm{d}\bar{t}\Big| \le \|\psi\| \int_G |K_N(\bar{t},\bar{s})|\,\mathrm{d}\bar{t} \le 2$$

but

$$\int_G \phi(\bar{t})\,\mu(\mathrm{d}\bar{t}) \in W$$

for all measurable ϕ with $|\phi(\bar{t})| \le 1$ and for all $\bar{t} \in G$. Therefore Φ_N is in $2W$ for all N and all ψ in $C(G)$ with $\|\psi\| \le 1$. That is, the set of Φ_N is weakly equi-compact.

Now suppose that the set of Φ_N is weakly equi-compact. Then, by definition, there exists a weakly compact subset H of X such that

$$\{\Phi_N(\psi);\ \psi \in C(G),\ \|\psi\| \le 1,\ N = 1,2,\dots\} \subset H\,.$$

Take x' in X'. Then there exists a constant $C_{x'}$ such that

$$|\langle \Phi_N(\psi),\, x'\rangle| \le C_{x'}$$

for all N and ψ with $\|\psi\| \le 1$. Therefore for each N

$$\sup_{\|\psi\|\le 1}\Big|\int_G \psi(\bar{t})\langle \bar{\sigma}_N(\bar{t}),\, x'\rangle\,\mathrm{d}\bar{t}\Big| \le C_{x'}$$

that is

$$\int_G |\langle \bar{\sigma}_N(\bar{t}),\, x'\rangle|\,\mathrm{d}\bar{t} \le C_{x'}\,.$$

From the scalar case (see [6, Theorem 1 (iii)]) it follows that there exists a scalar measure $\mu_{x'}$ such that

$$\langle a_n, x'\rangle = \int_G \psi_n(\bar{t})\,\mu_{x'}(\mathrm{d}\bar{t})\,. \tag{3}$$

Since there exists a regular, scalar Borel measure μ' such that, for all $\varepsilon > 0$ and all ψ in $C(G)$, there exists N such that

$$\left| \int_G \psi(\bar{t})\, \overline{\sigma}_N(\bar{t})\, \mathrm{d}\bar{t} - \int_G \psi(\bar{t})\, \mu'(\mathrm{d}\bar{t}) \right| < \varepsilon ,$$

for all $\psi \in C(G)$

$$\lim_N \langle \Phi_N(\psi),\, x' \rangle = \int_G \psi(\bar{t})\, \mu_{x'}(\mathrm{d}\bar{t}) . \tag{4}$$

This implies, for each fixed ψ, $\langle \Phi_N(\psi), x' \rangle$ is convergent for all $x' \in X'$. Since $\{\Phi_N(\psi);\ N = 1, 2, \dots\}$ is in the weakly compact set $\|\psi\| H$ and since it is weakly Cauchy, it is weakly convergent. Denote the weak limit by $\Phi(\psi)$. Then Φ is in H for all ψ with $\|\psi\| \le 1$. Since H is weakly compact, Φ is also weakly compact. By the theorem of Bartle, Dunford, Schwartz (see [4]), there exists a regular measure $\mu \colon \mathcal{B}(G) \to X$ such that

$$\Phi(\psi) = \int_G \psi(\bar{t})\, \mu(\,\mathrm{d}\bar{t})$$

for all $\psi \in C(G)$. By taking $\psi = \psi_n$ we have for all x' in X'

$$\big\langle \Phi(\psi_n)\,, x' \big\rangle = \int_G \psi_n(\bar{t}) \big\langle\, \mu(\mathrm{d}\bar{t}),\, x' \big\rangle .$$

But by (3) and (4)

$$\big\langle \Phi(\psi_n),\, x' \big\rangle = \int_G \psi_n(\bar{t})\, \mu_{x'}(\mathrm{d}\bar{t}) = \langle a_n,\, x' \rangle$$

hence

$$a_n = \int_G \psi_n(\bar{t})\, \mu(\mathrm{d}\bar{t})$$

and the proof is complete. □

Now we shall show how to isolate the discrete component of μ (m).

THEOREM 3.

1. If $\overline{S} = \overline{S}(d\mu)$, then the partial sums satisfy

$$\frac{\overline{S}_n(\bar{x})}{n} \to \mu\big(\{\bar{x}\}\big), \quad \bar{x} \in G .$$

2. If $S = S(dm)$, then

$$\frac{S_n(x)}{n} \to m\big(\{x\}\big), \quad x \in I .$$

P r o o f . We have

$$\frac{\overline{S}(\overline{x})}{n} = \int_G \frac{D_n(\overline{x},\overline{t})}{n}\,\mathrm{d}\mu(\overline{t}),$$

where D_n is the Dirichlet kernel. The integrand is bounded by 1 and converges to 1 at $\overline{t} = \overline{x}$ and to 0 elsewhere. The first assertion follows from Lebesgue's convergence theorem.

Let $\mu \sim m$. Applying Theorem 1(iv) gives

$$\frac{S_n(x)}{n} = \frac{\overline{S}_n(\rho(x))}{n} \to \mu(\{\rho(x)\}) = m(\{x\}).$$

□

THEOREM 4. *A Walsh series S on I is a Stieltjes series $S(dm)$ if and only if the following two conditions are satisfied:*

1. The set of maps $\Phi_N\colon C_W(I) \to X$, $N = 1,2,\dots$, defined by

$$\Phi_N(\psi) = \int_I \psi(t)\sigma_N(t)\,\mathrm{d}t, \qquad \psi \in C_W(I),$$

is weakly equi-compact.

2.

$$\frac{S_n(q-0)}{n} \to 0, \qquad q \in Q. \tag{5}$$

P r o o f . Let $S \sim \overline{S}$. By Fine's map $C(G) \sim C_W(I)$ and by Theorem 1 (i),

$$\int_G \psi(\overline{t})\overline{\sigma}_N(\overline{t})\,\mathrm{d}\overline{t} = \int_I \psi(t)\sigma_N(t)\,\mathrm{d}t.$$

So, by Theorem 2, the condition that (2) is weekly equi-compact is necessary and sufficient for $\overline{S} = \overline{S}(d\mu)$. Again by Theorem 1 ((iv) and (v)), $S = S(dm)$ is equivalent to μ being a usual measure and $\mu \sim m$. By Theorem 3, μ is usual if and only if

$$\frac{\overline{S}_n(\overline{q}')}{n} \to 0, \quad \overline{q}' \in \overline{Q}',$$

for every dyadic rational $\overline{q}$. $\overline{S}_n(\overline{q}') = S_n(q-0)$ and so the condition (5) holds true. □

We can relax the assumption 1 in Theorem 4. It is not difficult to verify that one can merely assume the weakly equi-compactness of the set of $\Phi_N(\psi)$ for all $\psi \in C(I)$.

THEOREM 5. *Given a sequence a_n, $n = 0,1,\dots$, of elements of X. There exists a regular measure $m\colon \mathfrak{B}(I) \to X$ of finite total variation such that a_n are the coefficients of m if and only if there exists a constant D such that*

1.

$$\int_I \| \sigma_N(t) \| \, \mathrm{d}t \le D, \quad N = 1, 2, \dots ,$$

where dt *denotes the Lebesgue measure on* I.

2.

$$\frac{S_n(q-0)}{n} \to 0, \quad q \in Q .$$

P r o o f. Let

1'.

$$\int_G \| \overline{\sigma}_N(\overline{t}) \| \, \mathrm{d}\overline{t} \le D, \quad N = 1, 2, \dots ,$$

where d$\overline{t}$ denotes the normalized Haar measure on G. We shall show that *1.* is equivalent to *1'.* At first we prove this theorem on G and then it is easy to return to the unit interval I. Suppose that there exists such measure μ on G. Then, for each N, by Proposition 2 ,

$$\int_G \| \overline{\sigma}_N(\overline{t}) \| \, \mathrm{d}\overline{t} = \int_G \| \int_G K_N(\overline{t}, \overline{s}) \, \mu(\mathrm{d}\overline{s}) \| \, \mathrm{d}\overline{t} \le$$

$$\le \int_G \Big(\int_G |K_N(\overline{t}, \overline{s})| \, \mathrm{d}\overline{t} \Big) \|\mu\|(\mathrm{d}\overline{s}) \le 2 \|\mu\| \overline{s}(G) .$$

Conversely, suppose that $\int_G \|\overline{\sigma}_N(\overline{t})\| \, \mathrm{d}\overline{t} \le D$ for all N. If we define

$$\Phi_N(\psi) = \int_G \psi(\overline{t}) \overline{\sigma}_N(\overline{t}) \mathrm{d}\overline{t}, \quad \psi \in C(G) ,$$

then $\|\Phi_N\| \le D$ for all N. For each n, $\lim_N \Phi_N(\psi_n) = a_n$ and then $\lim_N \Phi_N(\psi)$ exists for all ψ which are linear combinations of the characters ψ_n and so as $\|\Phi_N\| \le D$ for all N, we conclude that $\lim_N \Phi_N(\psi)$ exists for all ψ in $C(G)$. Denote this limit by $\Phi(\psi)$. For each subset A of G, let $C(G, A)$ denote the space of continuous functions on G vanishing outside A. Define for each A,

$$|||F_A||| = \sup \sum \|F(\psi_i)\| ,$$

where $F \colon C(G) \to X$ is a linear map and the supremum is taken over all finite families ψ_i in $C(G, A)$ with $\sum |\psi_i(\overline{t})| \le \chi_A(\overline{t})$ for all $\overline{t}$ in G.

To obtain required measure we use the following (see [5]): □

PROPOSITION 3. *If $F: C(G) \to X$ is a linear map, then there exists a regular measure $\mu: \mathfrak{B}(G) \to X$ with finite variation such that*

$$F(\psi) = \int_G \psi(\overline{t})\, \mu(\mathrm{d}\overline{t}), \quad \psi \in C(G),$$

if only if

$$|||F_A||| \leq \infty$$

for all A in $\mathfrak{B}(G)$.

Proof. Let A be in $\mathfrak{B}(G)$ and let $\{\psi_i;\ i = 1,2,\dots n\}$ be a finite family of functions in $C(G,A)$ with $\sum\limits_{i=1}^{n} |\psi_i(\overline{t})| \leq \chi_A(\overline{t})$, $\overline{t} \in G$. Then for each N,

$$\sum_{i=1}^{n} \|\Phi_N(\psi_i)\| = \sum_{i=1}^{n} \Big\| \int_G \psi_i(\overline{t}) \overline{\sigma}_N(\overline{t})\, \mathrm{d}\overline{t} \Big\| \leq$$

$$\leq \sum_{i=1}^{n} \int_G |\psi_i(\overline{t})|\ \|\overline{\sigma}_N(\overline{t})\| d\overline{t} \leq D.$$

Hence $\sum\limits_{1}^{n} \|\Phi(\psi_i)\| \leq D$ and so $|||\Phi_A||| \leq D$. By Proposition 3, there exists such a regular measure $\mu: \mathfrak{B}(G) \to X$ with finite variation that

$$\Phi(\psi) = \int_G \psi(\overline{t}) \mu(\mathrm{d}\overline{t}), \quad \psi \in C(G).$$

Since

$$\int_G \psi_n(\overline{t}) \overline{\sigma}_N(\overline{t})\, \mathrm{d}\overline{t} = \int_G \psi_n(\overline{t}) \sum_{j=0}^{N-1} \Big(\frac{N-n}{N}\Big) a_j \psi_j(\overline{t})\, \mathrm{d}\overline{t} =$$

$$= \Big(\frac{N-n}{N}\Big) a_n \to a_n, \quad N \to \infty, \tag{6}$$

if we substitute ψ by the characters ψ_n, we have

$$a_n = \int_G \psi_n(\overline{t}) \mu(\mathrm{d}\overline{t}),$$

and the proof of theorem is complete for the dyadic group G. To return to unit interval we use Theorem 1 (i). We have

$$\int_G \|\overline{\sigma}_N(\overline{t})\|\, \mathrm{d}\overline{t} = \int_I \|\sigma_N(t)\|\, \mathrm{d}t, \quad N = 1, 2, \dots,$$

so *1.* and *1'.* are equivalent and, by the proof on G, *1.* is necessary and sufficient for $\overline{S} = \overline{S}(d\mu)$. Again by Theorem 1 ((iv) and (v)), $S = S(dm)$ is equivalent to μ being a usual measure and $\mu \sim m$. By Theorem 3, the partial sums satisfy condition *2.* and the proof is finished. □

4. Fourier–Walsh expansion of X-valued functions

In this section, we will use the assertion of the following lemma.

LEMMA 2. *If $\overline{f}$ is in $L_p(G)$, $1 \le p < \infty$, then the linear maps $T_N\colon L_p \to L_p$ (for all p, $1 \le p \le \infty$) defined by*

$$(T_N\overline{f})(\overline{t}) = \int_G K_N(\overline{t},\overline{s})\overline{f}(\overline{s})\,\mathrm{d}\overline{s}, \quad \overline{f} \in L_p,$$

converge to $\overline{f}$ in the L_p-norm. (see [6])

THEOREM 6. *Given a sequence a_n, $n = 0,1,2,\dots$, of elements of X and a Pettis integrable function $f\colon \mathfrak{B}(I) \to X$, the a_n are the Fourier-Walsh coefficients of f if and only if*

$$\lim_N \int_I \psi(t)(\sigma_N(t) - f(t))\,\mathrm{d}t = 0 \tag{7}$$

for all ψ in $C_W(I)$ with $\|\psi\| \le 1$.

Proof. Let

$$\lim_N \int_G \psi(\overline{t})(\overline{\sigma}_N(\overline{t}) - \overline{f}(\overline{t}))\,\mathrm{d}\overline{t} = 0 \tag{7'}$$

for all ψ in $C(G)$ with $\|\psi\| \le 1$. Suppose that the a_n are the coefficients of $\overline{f}\colon \mathfrak{B}(G) \to X$. Let V be an absorbing neighbourhood of 0 in X. For all ψ in $C(G)$

$$\begin{aligned}\int_G \psi(\overline{t})(\overline{\sigma}_N(\overline{t}) - \overline{f}(\overline{t}))\,\mathrm{d}\overline{t} &= \int_G \psi(\overline{t})\Big(\int_G K_N(\overline{t},\overline{s})(\overline{f}(\overline{s})\,\mathrm{d}\overline{s} - \overline{f}(\overline{t})\Big)\,\mathrm{d}\overline{t}\\ &= \int_G \overline{f}(\overline{s})\Big(\int_G K_N(\overline{t},\overline{s})\psi(\overline{t})\,\mathrm{d}\overline{t} - \psi(\overline{s})\Big)\,\mathrm{d}\overline{s}.\end{aligned}$$

There exists a constant $\varepsilon > 0$ such that for all γ with $|\gamma| < \varepsilon$

$$\int_G \gamma\overline{f}(\overline{s})\,\mathrm{d}\overline{s} \in V.$$

Since there is an integer $N_0(\psi)$ such that, for all $N \ge N_0(\psi)$

$$\Big|\int_G K_N(\overline{t},\overline{s})\,\psi(\overline{t})\,\mathrm{d}\overline{t} - \psi(\overline{s})\Big| < \varepsilon$$

for each ψ in $C(G)$ with $\|\psi\| \le 1$ we have that

$$\int_G \psi(\bar{t})\big(\bar{\sigma}_N(\bar{t}) - \bar{f}(\bar{t})\big)\, \mathrm{d}\bar{t} \in V$$

if $\psi \in C(G)$ with $\|\psi\| \le 1$, $N \ge N_0(\psi)$.

Conversely, let us suppose that (7') holds true. Put

$$\Phi_{N(\psi)} = \int_G \psi(\bar{t})\bar{\sigma}_N(\bar{t})\, \mathrm{d}\bar{t}, \qquad N = 1, 2, \dots ,$$

and put

$$\Phi(\psi) = \int_G \psi(\bar{t})\bar{f}(\bar{t})\, \mathrm{d}\bar{t}, \qquad \psi \in C(G).$$

(7') implies $\lim\limits_N \Phi_N(\psi) = \Phi(\psi)$ for all ψ in $C(G)$ with $\|\psi\| \le 1$. Then for all $x' \in X'$

$$\lim_N \langle \Phi_N(\psi),\, x'\rangle = \langle \Phi(\psi),\, x'\rangle .$$

Hence for every x' in X'

$$\langle a_n, x'\rangle = \lim_N \frac{N-n}{N}\langle a_n,\, x'\rangle = \lim_N \int_G \psi_n(\bar{t})\langle \bar{\sigma}_N(\bar{t}),\, x'\rangle \mathrm{d}\bar{t}$$

$$= \Big\langle \int_G \psi_n(\bar{t})\bar{f}(\bar{t})\, \mathrm{d}\bar{t}, x' \Big\rangle .$$

and so

$$a_n = \int_G \psi_n(\bar{t})\bar{f}(\bar{t})\, \mathrm{d}\bar{t} .$$

The proof is complete for dyadic group G.

Let $S \sim \overline{S}$. Since $C_W(I) \sim C(G)$, $f \sim \bar{f}$, the normalized Haar measure vanishes on $\overline{Q}'$ and Lebesgue measure does on Q', so, by Theorem 1 (i) and (vi),

$$\lim_N \int_G \psi_n(\bar{t})\big(\bar{\sigma}_N(\bar{t}) - \bar{f}(\bar{t})\big)\, \mathrm{d}\bar{t} = \lim_N \int_I \psi_n(t)\big(\sigma(t) - f(t)\big)\, \mathrm{d}t .$$

Then $f \sim \bar{f}$ is equivalent to $S = S(f)$ and the proof is finished. □

Similarly as in the comment after the Theorem 4, we can weaken (7) by requiring the convergence only for $\psi \in C(I)$.

Let X be a Banach space.

THEOREM 7. *Given a sequence a_n, $n = 1,2,\dots$, of elements of X, there exists an X-valued Bochner integrable function $f\colon \mathfrak{B}(I) \to X$ on I such that the a_n's are the coefficients of f if and only if*

$$\lim_{N,J\to\infty} \int_I \|\sigma_N(t) - \sigma_J(t)\| \,\mathrm{d}t = 0 . \tag{8}$$

P r o o f. Let

$$\lim_{N,J\to\infty} \int_G \|\overline{\sigma}_N(\overline{t}) - \overline{\sigma}_J(\overline{t})\| \,\mathrm{d}\overline{t} = 0 . \tag{8'}$$

Similarly as in previous theorem, (8) is equivalent to (8'). Suppose that $\overline{f}$ is Bochner-integrable and the a_n's are the Walsh-Fourier coefficients of f. Let $\{G_i\}_1^n$ be a finite family of vectors in X and define $g\colon G \to X$ by

$$g(\overline{t}) = \sum_{i=1}^{n} \beta_i \chi_{G_i}(\overline{t}) .$$

Then

$$\int_G \Big\| \int_G K_N(\overline{t},\overline{s}) g(\overline{s}) \mathrm{d}\overline{s} - g(\overline{t}) \Big\| \mathrm{d}\overline{t} = \int_G \Big\| \sum_{i=1}^{n} \beta_i \Big(\int_G K_N(\overline{t},\overline{s}) \chi_{G_i}(\overline{s}) \mathrm{d}\overline{s} - \chi_{G_i}(\overline{t}) \Big) \Big\| \mathrm{d}\overline{t} \le$$

$$\le \sum_{i=1}^{n} \Big(\|\beta_i\| \int_G \Big| \int_G K_N(\overline{t},\overline{s}) \chi_{G_i}(\overline{s}) \,\mathrm{d}\overline{s} - \chi_{G_i}(\overline{t}) \Big| \,\mathrm{d}\overline{t} \Big)$$

which, by Lemma 2, tends to 0 as $N \to \infty$. Since the set of all such g's is dense in the space of all Bochner integrable functions and since $\overline{\sigma}_N(\overline{t}) = \int_G K_N(\overline{t},\overline{s}) f(\overline{s}) \,\mathrm{d}\overline{s}$, we have

$$\lim_N \int_G \|\overline{\sigma}_N(\overline{t}) - f(\overline{t})\| \,\mathrm{d}\overline{t} = 0 .$$

Conversely, suppose that $\{\sigma_N\}$ is a Cauchy sequence in the norm of the space of all Bochner integrable functions. Since this space is a Banach space, σ_N converges in the Bochner space norm to a Bochner integrable function f. Hence

$$\Big\| \int_G \big(f(\overline{t}) \overline{\sigma}_N(\overline{t}) \big) \psi_n(\overline{t}) \,\mathrm{d}\overline{t} \Big\| \le \int_G \|f(\overline{t}) - \overline{\sigma}_N(\overline{t})\| \, |\psi_n(\overline{t})| \,\mathrm{d}\overline{t} \le$$

$$\le \sup_{\overline{t}} |\psi_n(\overline{t})| \, \|\sigma_N - f\|_B \to 0, \quad N \to \infty .$$

Thus, by (6),

$$a_n = \int_G \psi_n(\overline{t}) f(\overline{t}) \,\mathrm{d}\overline{t} .$$

Put $f(x) = \overline{f}(\rho(x))$. By returning to unit interval I, it is easy to see that f is Bochner integrable if and only if $\overline{f}$ is Bochner integrable, since $f \sim \overline{f}$. By Theorem 1 (i),

$$\lim_{N,J\to\infty} \int_G \|\overline{\sigma}_N(\overline{t}) - \sigma_J(\overline{t})\| \, d\overline{t} = \lim_{N,J\to\infty} \int_I \|\sigma_N(t) - \sigma_J(t)\| \, dt .$$

Since $f \sim \overline{f}$, Haar measure vanishes on $\overline{Q}'$, so $S = S(f)$. □

REFERENCES

[1] FINE, N. J.: *Fourier-Stieltjes series of Walsh functions*, Trans. Amer. Math. Soc. **86** (1957), 246–255.

[2] SCHIPP, F.—WADE, W. R.—SIMONP.: *Walsh Series*, Akadémiai Kiadó, Budapest, 1990.

[3] TWEDDLE, I.: *Weak compactness in locally convex spaces*, Glasgow Math. J. **9** (1968), 123–127.

[4] KLUVÁNEK, I.: *Fourier transforms of vector-valued functions and measures*, Studia Math. **37** (1970), 1–12.

[5] DUNFORD, N.—SCHWARTZ, J. T.: *Linear Operators, Part I.*, Interscience, New York, 1966.

[6] Mc KEE, S.: *Orthogonal expansion of vector-valued functions and measures*, Matematický časopis **22** (1972).

Received April 22, 1993

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