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ANOTHER CHARACTERIZATION OF THE ORTHOMODULARITY OF THE ORTHOLATTICE OF ALL POLARS

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ABSTRACT. Let $\perp$ be a symmetric, irreflexive relation on a set A and put $B^\perp := \{a \in A | a \perp b$ for all $b \in B\}$ for all $B \subseteq A$. Then $L := (\{B^\perp | B \subseteq A\}, \subseteq, ^\perp, \emptyset, A)$ is an ortholattice. Simple conditions on $\perp$ are given which are equivalent to the orthomodularity of L.

At the Second Winter School on Measure Theory, R. Zapatrin posed the following problem:

Let $(A, \perp)$ be a set with a symmetric, irreflexive relation $\perp$. For $B \subseteq A$ the map $B^\perp = \{a \in A \colon a \perp b$, for all $b \in B\}$ is called the *polar* of B. The lattice of all polars is always an ortholattice. Under what conditions on $\perp$ is this ortholattice orthomodular?

In [1], it was proved that it is not possible to solve this problem without using quantification over subsets of A. In [2, p. 260], e.g., the following (rather complicated) condition was proved to be equivalent to the orthomodularity of the ortholattice of all polars: For all subsets B of A and all $a \in A$ for which there exists a $b \in A$ with both $b \perp c$ for all $c \in B$ and $b \not\perp a$ there exists a $d \in A$ with (1) and (2):

(1) $d \perp e$ for all $e \in B$.

(2) If $f \in A$ and $f \perp g$ for all $g \in B \cup \{a\}$ then $d \perp f$.

The aim of this note is to give another simpler characterization of the orthomodularity of the ortholattice of all polars.

THEOREM. *The ortholattice of all polars is orthomodular iff there do not exist subsets B, C of A with (i) – (iii):*

(i) $B \subset C$.

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(ii) *There exists an* $a \in A$ *such that* $a \perp b$ *for all* $b \in B$ *but* $a \not\perp c$ *for at least one* $c \in C$.

(iii) *If* $a \in A$ *and* $a \perp b$ *for all* $b \in B$ *then there exists a* $d \in A$ *with* $d \perp c$ *for all* $c \in C$ *but* $d \not\perp a$.

Proof. Suppose first, there exist such subsets B, C of A. Then $C^\perp \subseteq B^\perp$ but $C^\perp \vee (B^\perp \wedge C^{\perp\perp}) = C^\perp \vee (B^\perp \cap C^{\perp\perp}) = C^\perp \vee \emptyset = C^\perp \subset B^\perp$ and hence the ortholattice of all polars is not orthomodular. Conversely, assume the ortholattice of all polars not to be orthomodular. Then there exist subsets D, E of A with $D^\perp \subseteq E^\perp$ and $D^\perp \vee (E^\perp \wedge D^{\perp\perp}) \subset E^\perp$. Put $F := \big(D^\perp \vee (E^\perp \wedge D^{\perp\perp})\big)^\perp$ and $G := E \cup F$. Then $G^\perp = E^\perp \cap F^\perp = F^\perp$. Because of $F^\perp \subset E^\perp$ we have $F \not\subseteq E$ which implies $E \subset G$. Moreover, we have $G^\perp \subset E^\perp$ and $E^\perp \cap G^{\perp\perp} = E^\perp \wedge \big(D^\perp \vee (E^\perp \wedge D^{\perp\perp})\big)^\perp = E^\perp \wedge D^{\perp\perp} \wedge (E^{\perp\perp} \vee D^\perp) = E^\perp \wedge D^{\perp\perp} \wedge (E^\perp \wedge D^{\perp\perp})^\perp = \emptyset$. Hence (i) – (iii) (with E, G instead of B, C) are satisfied. This completes the proof. □

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