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STATES ON PROJECTION LOGICS OF OPERATOR ALGEBRAS

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ABSTRACT. We summarize some recent results concerning measures on projection lattices of operator algebras. Especially we pursue extension properties of operator algebras and description of states defined on projections.

1. Introduction and preliminaries

The structure of projections in operator algebra is one of the most important examples of quantum logics as regards application in quantum physics [12, 19, 21, 22, 25, 27, 30]. Projection logics have inspired the theory of orthomodular structures [18] and has been of particular importance for the theory of operator algebras itself.

Our results have been motivated by the position of projection logics in the general context of orthomodular structures (part 1) as well as by concrete questions of the noncommutative measure theory on projections (part 2). Before formulating our results, let us recall a few notions and fix notation. (For the general theory of quantum logic we refer to [27], for the theory of operator algebras we refer to [17, 23, 33]).

By a *(quantum) logic* we mean a partially ordered set $(L, \leq)$ with an orthocomplementation operation, $\perp$, satisfying the following conditions $(a, b \in L)$:

(i) L has a least and a greatest element 0_L and 1_L, respectively;
(ii) $a \leq b$ implies $b^\perp \leq a^\perp$;
(iii) $a = a^{\perp\perp}$;
(iv) if $a \leq b^\perp$, then the supremum $a \vee b^\perp$ exists in L;
(v) if $a \leq b$, then $b = a \vee (b \wedge a^\perp)$.

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Elements $a, b \in L$ are said to be *orthogonal* (in symbol $a \perp b$) if $a \leq b^{\perp}$. A logic K is a sublogic of a logic L if $K \subset L$ and if the ordering, the greatest element, the least element, the orthocomplementation operation and the formation of the suprema of orthogonal elements coincide for K and L.

A complex-valued function m on a logic L is said to be a *measure* if $m(a \vee b) = m(a) + m(b)$ whenever $a \perp b$ $(a, b \in L)$. By a *state* (*probability measure*) of L we mean a measure s on L which is positive and $s(1_L) = 1$. The *state space* $S(L)$ of L is the set of all states of L. A state $s \in S(L)$ is called *pure* if it is an extreme point of $S(L)$. A logic L is called *unital* if for every nonzero $a \in L$ there is a state $s \in S(L)$ such that $s(a) = 1$.

Throughout the paper, let M be a unital C^*-algebra (resp. unital JB algebra) with the unit I. By a projection in M we mean a self-adjoint idempotent, or an idempotent, if M is a C^*-algebra or a JB algebra, respectively. Let $P(M)$ be the set of all projections in M. When we endow $P(M)$ with the partial ordering defined by the positive cone of M, then $P(M)$ becomes a logic (orthocomplementation is given by $p^{\perp} = I - p$). We say that $P(M)$ is the *projection logic* of M.

2. Universal state extension property

The central notion of this section is the following property of states. Let L be a unital logic. We say that L has the *universal state extension property* if the following condition is satisfied: Let L be a sublogic of a unital logic K. Then every state of L extends to a state of K. Following [10, 11, 31, 32] we say that a C^*-algebra M has the *Gleason property* if every state on the projection logic $P(M)$ extends to a linear state (=positive normalized functional) on M. Let us first observe that M has the Gleason property whenever $P(M)$ has the universal state extension property. Indeed, if M is viewed as a unital subalgebra of the type I_n von Neumann factor N, where $n \geq 3$, then $P(M)$ is a sublogic of $P(N)$. Suppose that $P(M)$ has the universal state extension property. Then every state ϱ of P(M) can be extended to a state $\hat{\varrho}$ on $P(N)$. According to [10, 11, 31, 32], $\hat{\varrho}$ extends to a linear state of N. Conversely, the main result of this section says that for important class of C^*-algebras the Gleason property already implies the universal state extension property.

THEOREM 2.1 ([14]). *Let M be a unital C^*-algebra such that every maximal abelian subalgebra of M is the norm closed linear span of its projections. If M has the Gleason property, then $P(M)$ has the universal state extension property.*

As a corollary of the previous theorem we obtain, e.g., that every projection logic of a von Neumann algebra not containing type I_2 direct summand has the universal state extension property. This generalizes hitherto known results about universal state extension property of Boolean algebras and Hilbert-space logics [16, 26]. In physical interpretation Theorem 2.1. means that any enlargement of the quantum system given by a von Neumann algebra formalism imposes no restriction on its physically admissible states.

COROLLARY 2.2 ([14]). *Let M be a von Neumann algebra not containing type I_2 direct summand. Let J be a norm closed (two-sided) ideal in M. Then the projection logics of both algebras $J+I$ and M/J have the universal state extension property.*

By further analysis it can be shown that the extensions guaranteed by Theorem 2.1 can by taken always linear in some linear structure associated with a larger logic. To this aim let us introduce the following notions. Let L be a logic with the nonempty state space $S(L)$. Let us denote by $A^b(L)$ the real Banach space of all bounded real affine functions on $S(L)$ with supremum norm. In the sequel we will follow the general theory of order unit norm spaces (see, e.g., [3, 17, 29]). Endowed with the ordering $f \leq g \Leftrightarrow f(s) \leq g(s)$ for all $s \in S(L)$, the space $A^b(L)$ forms a complete order unit norm space with the order unit u_L (the constant function on $S(L)$ equals 1). A linear functional ϱ on $A^b(L)$ is called a *state* if it is positive and if $\varrho(u_L) = 1$. Let e_L denote the canonical evaluation mapping of L into $A^b(L)$ given by the formula $e_L(a)(s) = s(a)$ for all $a \in L$ and $s \in S(L)$. (See, e.g., [6, 20, 28] for the convex theory of state spaces of quantum logics.)

THEOREM 2.3 ([14]). *Let M be a unital C^*-algebra with the Gleason property and let every maximal abelian subalgebra of M be a norm closed linear span of the projections. Let L be a unital logic containing $P(M)$ as a sublogic. Then every state of $P(M)$ extends to a state of L. Moreover, the latter state is of the form $f \circ e_L$, where f is a state of $A^b(L)$.*

According to Theorem 2.3, every state of $P(M)$ can be viewed (upon an obvious identification) as a restriction of the linear state of $A^b(L)$. Analogously we can reformulate Corollary 2.2, too.

Let us remark in the conclusion of this section that its results can be generalized by using [9] for all bounded measures with values in finite-dimensional spaces.

3. Commutative properties of states

The second part of the paper deals with various types of states on von Neumann algebras and JBW algebras. We provide a characterization of some properties of states familiar within classical measure theory. It turns out that some concepts of commutative measure theory have rather unexpected counterpart in the noncommutative framework. All results in this section were proved jointly by L. J. Bunce and the author (see [4, 5]).

At first we shall review some results on Jauch–Piron states. Let M be a von Neumann algebra. A state ϱ of $P(M)$ is said to be *Jauch–Piron* if $\varrho(e \vee f) = 0$ whenever $e, f \in P(M)$ with $\varrho(e) = \varrho(f) = 0$ (see, e.g., [19] for a physical explanation of this concept). We say that a state ϱ of $P(M)$ is *σ-additive*, if $\varrho\Big(\sum\limits_{i=1}^{\infty} p_n\Big) = \sum\limits_{i=1}^{\infty} \varrho(p_n)$, whenever (p_i) is a sequence of orthogonal projections in M.

THEOREM 3.1 ([4]). *Let M be a von Neumann algebra without abelian and type I_2 part. A pure state of $P(M)$ is Jauch-Piron if and only if it is σ-additive.*

A more detailed description of pure Jauch-Piron state is discussed in [2, 15]. A state ϱ of $P(M)$ is said to be *regular* if $\varrho\Big(\sum\limits_{n=1}^{\infty} e_n\Big) = 0$ whenever $(e_n) \subset P(M)$ with $\varrho(e_n) = 0$ for all $n \in \mathbb{N}$. It turns out that a pure state is σ-additive if and only if it is regular. Using regularity instead of σ-additivity we can generalize Theorem 3.1. in the following way:

THEOREM 3.2 ([4]). *Let M be a von Neumann algebra without type I_2 part satisfying one of the following conditions:*

(i) *M is a factor.*
(ii) *M has no locally σ-finite direct summand.*
(iii) *M is σ-finite type III.*

Then a state ϱ of $P(M)$ is Jauch–Piron if and only if it is regular. Moreover, if M is σ-finite then ϱ is Jauch–Piron if and only if ϱ has a support.

As can be demonstrated by examples, these results cannot be extended to all von Neumann algebras. Nevertheless, there exists an intimate relation between Jauch–Piron property and non-singularity of states on centers of hereditary subalgebras (for details see [4]).

In the conclusion of the paper we shall consider subadditive states on JBW algebras. Let M be a JBW algebra. A state ϱ of $P(M)$ is said to be *subadditive* if $\varrho(e \vee f) \le \varrho(e) + \varrho(f)$ for all $e, f \in P(M)$. Let us observe that subadditivity can be viewed as a stronger form of the Jauch–Piron property. While all states on abelian algebras are trivially subadditive it turns out that the only subadditive states on general $P(M)$ are tracial states. We recall that a state ϱ of M is said to be *tracial* if $\varrho\big(U_x(y^2)\big) = \varrho\big(U_y(x^2)\big)$ for all $x, y \in M$, where $U_x : M \to M$ is the map defined by $U_x(y) = 2x \circ (x \circ y) - x^2 \circ y$.

THEOREM 3.3 ([5]). *Let M be a JBW algebra and let ϱ be a subadditive state on $P(M)$. Then ϱ extends uniquely to a tracial state on M.*

An analogous result can be derived also for completely additive semifinite positive measures. We recall that a measure $\varrho\colon P(M) \to [0,\infty]$ is semifinite, if for each projection e there is a net of projections $e_\alpha \nearrow e$ with $\varrho(e_\alpha) < \infty$, for all α.

THEOREM 3.4 ([5]). *Let M be a JBW algebra and ϱ a subadditive completely additive semifinite measure on M. Then M extends uniquely to a normal semifinite trace on M.*

Traces in JBW algebras have been extensively studied in literature [1, 13, 24] and, in particular, they are important for the structure theory of JBW as well as von Neumann algebras. Our results provide purely lattice and measure theoretical characterization of traces on JBW algebras.

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