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YOSIDA–HEWITT DECOMPOSITIONS OF RIESZ SPACE–VALUED MEASURES ON ORTHOALGEBRAS

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ABSTRACT. We prove generalizations of the Yosida-Hewitt decomposition theorem for positive finitely additive measures defined on orthoalgebras (generalizing Boolean algebras and orthomodular posets = quantum logics) with values in a Dedekind complete Riesz space.

1. Introduction

The classical result of Yosida–Hewitt [15] has received attention of many authors [1, 4, 14, 2, 5, 6, 3] studying finitely additive measures on orthomodular posets. In [4, 3], Yosida-Hewitt-type decomposition theorems for Dedekind's complete normed Riesz space-valued measures have been presented.

In the last years, axiomatic models describing the propositional system of quantum mechanics are very important. Such are quantum logics (= orthomodular posets), or, more generally, orthoalgebras, originally introduced by Randall and Foulis [12, 13].

In the present note, we generalize the Yosida-Hewitt decomposition theorems for Riesz space-valued measures on orthoalgebras. These decompositions generalize those ones from [3, 4].

2. Orthomodular posets

An *orthomodular poset* (OMP) is a partially ordered set L with an ordering $\leq$, the smallest and greatest elements 0 and 1, respectively, and an orthocom-

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plementation $\perp\colon L \to L$ such that

(i) $a^{\perp\perp} = a$ for any $a \in L$;
(ii) $a \vee a^{\perp} = 1$ for any $a \in L$;
(iii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$;
(iv) if $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$;
(v) if $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ (orthomodular law).

We recall that from the above axioms we have de Morgan laws

$$\Big(\bigvee_i a_i\Big)^{\perp} = \bigwedge_i a_i^{\perp} \quad \text{and} \quad \Big(\bigwedge_i a_i\Big)^{\perp} = \Big(\bigvee_i a_i^{\perp}\Big) \tag{3.1}$$

saying that if one side of an equality exists in L, so exists the second one, and both are equal. If in an orthomodular poset L the join of any sequence (any system) of mutually orthogonal elements exists, we say that L is a *σ-orthomodular poset* (a *complete orthomodular poset*). An *orthomodular lattice* is an orthomodular poset L such that, for any $a, b \in L$, $a \vee b$ exists in L (using de Morgan laws, $a \wedge b$ exists in L, too). A distributive orthomodular lattice is called a *Boolean algebra.* We recall that an orthomodular lattice L is a Boolean algebra iff for any pair $a, b \in L$ there are three mutually orthogonal elements $a_1, b_1, c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. For more details concerning orthomodular posets and lattices see, e.g. [9, 11].

One of the most important cases of orthomodular lattices is the system of all closed subspaces, $L(H)$, of a real or complex Hilbert space H, with an inner product $(\cdot,\cdot)$. Here the partial ordering, $\leq$, is induced by the natural set-theoretic inclusion, and $M^{\perp} = \{x \in H\colon\ (x,y) = 0 \text{ for any } y \in M\}$. Then $L(H)$ is a complete orthomodular lattice, which is not a Boolean algebra, if $\dim H \neq 1$.

If S is an inner product space (not necessarily complete), denote by $E(S)$ the set of all *splitting subspaces* of S, i.e., the set of all $M \subseteq S$ such that $M + M^{\perp} = S$. Then $E(S)$ is an orthomodular poset which is not necessarily a σ-orthomodular poset. We recall that according to [7], S is complete if and only if $E(S)$ is a σ-orthomodular poset.

3. Orthoalgebras

An *orthoalgebra* is a set L with two particular elements 0, 1, and with a partial binary operation $\oplus\colon\ L \times L \to L$ such that for all $a, b, c \in L$ we have

(i) if $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$;
(ii) if $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(iii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$;

(iv) if $a \oplus a$ is defined, then $a = 0$.

If the assumptions of (ii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less or equal* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^{\perp}$.

In [8], there are proofs of the following statements:

PROPOSITION 3.1. *Let a, b and c be elements of an orthoalgebra L. Then*

(i) $a \perp b \Leftrightarrow b \perp a$.

(ii) $a \perp a \Rightarrow a = 0$.

(iii) $a \perp 1 \Leftrightarrow a = 0$.

(iv) $a^{\perp\perp} = a$.

(v) $1^{\perp} = 0$ *and* $0^{\perp} = 1$.

(vi) $a \perp b \Rightarrow a \perp (a \oplus b)^{\perp}$, $a \oplus (a \oplus b)^{\perp} = b^{\perp}$.

(vii) $a \perp b \Leftrightarrow a \leq b^{\perp}$.

(viii) $a \leq b \Rightarrow b = a \oplus (a \oplus b^{\perp})^{\perp}$.

(ix) $a \oplus b = a \oplus c \Rightarrow b = c$.

(x) $a \oplus b \leq a \oplus c \Rightarrow b \leq c$.

(xi) $0 \leq a \leq 1$, *and* $\leq$ *is a partial ordering on* L.

(xii) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.

(xiii) $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = 1$.

(xiv) $a \perp b$, $a \vee b \in L \Rightarrow a \oplus b = a \vee b$.

We see that if L is an orthomodular poset and $a \oplus b := a \vee b$ whenever $a \perp b$ in L, then L with $0, 1, \oplus$ is an orthoalgebra. The converse statement does not hold, in general, as it follows from the example of R. Wright [8]:

EXAMPLE 3.2. Let $L = \{0, 1, a, b, c, e, f, a^{\perp}, b^{\perp}, c^{\perp}, d^{\perp}, e^{\perp}, f^{\perp}\}$ with $a \oplus b = d \oplus e = c^{\perp}$, $b \oplus c = e \oplus f = a^{\perp}$, $c \oplus d = f \oplus a = e^{\perp}$, $c \oplus e = d^{\perp}$, $a \oplus c = b^{\perp}$, $e \oplus a = f^{\perp}$ is an orthoalgebra that is not an orthomodular poset.

We recall that an orthoalgebra L is an OMP iff $a \perp b$ implies $a \vee b \in L$.

For $a, b \in L$ with $a \leq b$, we define the *difference* of a in b as the unique element c in L such that $a \oplus c = b$, and we write $c = b \ominus a$. It is evident that $b \ominus a = (a \oplus b^{\perp})^{\perp}$.

4. Riesz spaces

Let V be a real vector space with a partial ordering $\leq$ such that

(i) if $x, y \in V$, then $x \wedge y \in V$, $z \vee y \in V$;
(ii) if $x \leq y$, then $x + z \leq y + z$, for any $z \in V$;
(iii) if $x \leq y$, then $\alpha x \leq \alpha y$ for any $\alpha \in \mathbb{R}_+$,

then V is said to be a *Riesz space.* We define for any $x \in V$: $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x^+ + x^-$. We have for all $x, y \in L$ (i) $x = x^+ - x^-$; (ii) $|x| = 0$ iff $x = 0$; (iii) $|x+y| \leq |x| + |y|$. By V_+ we denote the set of all positive elements of V, i.e., $V_+ = \{x \in V : x \geq 0\}$.

It is well-known that V is a distributive lattice, where the following equalities hold:

$$a + \bigvee_i a_i = \bigvee_i (a + a_i), \quad a + \bigwedge_i a_i = \bigwedge_i (a + a_i), \tag{4.1}$$

$$a - \bigvee_i a_i = \bigwedge_i (a - a_i), \quad a - \bigwedge_i a_i = \bigvee_i (a - a_i), \tag{4.2}$$

providing that if one side of above equalities exists in V, so exists the second one, and both coincide.

A Riesz space V is said to be *Dedekind complete* if, for any non-void majorized subset B of V, $\bigvee B := \bigvee\{b : b \in B\}$ exists in V.

A non-empty set D of V is *directed downwards* (*upwards*), and we write $D \downarrow$ ($D \uparrow$), if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$, $z \leq y$ ($z \geq x$, $z \geq y$). Two downwards directed sets $\{x_t : t \in T\}$ and $\{y_t : t \in T\}$ indexed by the same index set T are called *equidirected* if, for any $s, t \in T$, there exists $v \in T$ such that $x_v \leq x_s$ and $x_v \leq x_t$ as well as $y_v \leq y_s$ and $y_v \leq y_t$. A similar definition holds for upwards directed sets.

Let $x \in V$ and $D \subset V$. We say that $D \uparrow x$ if $D \uparrow$ and $x = \bigvee D$. Dually we define $D \downarrow x$., i.e., $D \downarrow$ and $x = \bigwedge D$. If $\{f_t\}$ and $\{g_t\}$ are equidirected, then [10, Theorem 15.8]:

$$\{f_t\} \uparrow f,\ \{g_t\} \uparrow g \implies \{f_t + g_t\} \uparrow f + g, \tag{4.3}$$

$$\{f_t\} \downarrow f,\ \{g_t\} \downarrow g \implies \{f_t + g_t\} \downarrow f + g. \tag{4.4}$$

5. Measures on orthoalgebras

Throughout this paper by L we understand an orthoalgebra, and V is a Dedekind complete Riesz space. Define the following natural ordering $\leq_n$ on V^L: $\mu_1 \leq_n \mu_2$ iff $\mu_1(a) \leq \mu_2(a)$ for any $a \in L$.

We say that an element $\mu \in V^L$ is a *finitely additive measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a \oplus b$ is defined in L. Then $\mu(0) = 0$, and $\mu(a^\perp) = \mu(1) - \mu(a)$, $a \in L$. If $\mu\colon L \to V_+$, then $a \le b$ implies $\mu(a) \le \mu(b)$.

To define σ-additive and completely additive measures on L, we introduce the following notions.

Let $F = \{a_1, \dots, a_n\} \subseteq L$. Recursively we define for $n \ge 3$

$$a_1 \oplus \cdots \oplus a_n \colon = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n\,, \tag{5.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in orthoalgebras we conclude that (5.1) is correctly defined. Definitorically we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$ and any k with $1 \le k \le n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n}\,, \tag{5.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n)\,. \tag{5.3}$$

We say that a finite set $F = \{a_1, \dots, a_n\}$ of L is $\bigoplus$*-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has $\bigoplus$*-sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^n a_i = a_1 \oplus \cdots \oplus a_n\,. \tag{5.4}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary subset G of L is $\bigoplus$*-orthogonal* if every finite subset F of G is $\bigoplus$-orthogonal. If G is $\bigoplus$-orthogonal, so is any its subset. An $\bigoplus$-orthogonal subset $G = \{a_i\colon\ i \in I\}$ of L has $\bigoplus$*-sum* in L, written as $\bigoplus_{i \in I} a_i$, if in L there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_F \bigoplus_{i \in F} a_i\,, \tag{5.5}$$

where F runs over all finite subsets in I. In this case, we also write $\bigoplus G := \bigoplus_{i \in I} a_i$.

It is evident that if $G = \{a_1, \dots, a_n\}$ is $\bigoplus$-orthogonal, then the $\bigoplus$-sum defined by (5.4) and (5.5) coincide.

We say that an orthoalgebra L is a *complete orthoalgebra* (*σ-orthoalgebra*) if, for any $\bigoplus$-orthogonal subset (any countable $\bigoplus$-orthogonal subset) G of L, there exists the $\bigoplus$-join in L. It is straightforward to verify that an orthoalgebra L is a σ-orthoalgebra if, for any sequence $\{a_i\}$ in L with $a_1 \le a_2 \le \cdots$, the join $\bigvee_{i=1}^{\infty} a_i$ exists in L. In addition, the following statement holds:

PROPOSITION 5.1. (1) *If L is a complete orthoalgebra, then any chain C in L has the join $\bigvee C$ in L.*

(2) *If L is an orthoalgebra such that any upwards directed system $D \subseteq L$ has a join in L,*[1] *then L is a complete orthoalgebra.*

Proof. (1) Let C be a chain in L. Denote by D the set of all possible differences $b \ominus a$, where $a \le b$, and $a, b \in C \cup \{0\}$. Since $a = a \ominus 0$, it follows that $C \subseteq D$. We claim that D is an $\bigoplus$-orthogonal family in L. Indeed, let $d_1, \dots, d_n \in D$ be given. Then $d_i = b_i \ominus a_i$, where $a_i \le b_i$, $a_i, b_i \in C \cup \{0\}$. Therefore, there exists a set $\{d_1^*, \dots, d_{2n}^*\} \subseteq C$ such that $\{a_1, \dots, a_n, b_1, \dots, b_n\} = \{d_1^*, \dots, d_{2n}^*\}$, and $d_1^* \le d_2^* \le \cdots \le d_{2n}^*$.

Put $e_i = d_i^* \ominus d_{i-1}^*$, $i = 1, \dots, 2n$, where $d_0 := 0$. Then, for any k with $1 \le k \le 2n$, we have

$$d_k^* = e_1 \oplus \cdots \oplus e_k \in L,$$

and, if $1 \le j < k \le 2n$, then

$$d_k^* \ominus d_j^* = e_{j+1} \oplus \cdots \oplus e_k . \tag{5.6}$$

Consequently, for any $d_i = b_i \ominus a_i$, $i = 1, \dots n$, there exists a finite subset F_i of $\{1, \dots, 2n\}$ such that $d_i = \bigoplus_{j \in F_i} e_j$, which by (5.6) implies $\bigoplus_{i=1}^{n} d_i \in L$.

Since L is a complete orthoalgebra, there exists $a_0 = \bigoplus D$. Because $C \subseteq D$, we have, for any $a \in C$, $a \le a_0$. Now, for some $c \in L$, let $a \le c$ for any $a \in C$. Then for all $a, b \in C$ with $a \le b$ we have $b \ominus a \le b \le c$. Therefore, from the first part of the present proof, we conclude

$$\bigoplus_{i=1}^{n} b_i \ominus a_i \le d_{2n}^* \ominus d_0^* = d_{2n}^* \le c,$$

which means $a_0 \le c$, and, finally, $\bigvee C = a_0$.

(2) The second statement follows easily from the observation, that if $\{a_i \colon i \in I\}$ is an $\bigoplus$-orthogonal set in L, then $\{b_F := \bigoplus_{i \in F} a_i \colon F$ is a finite subset of $I\}$ is an upwards directed family in L having a join in L. □

[1] See the same definition as that for Riesz spaces.

A mapping $\mu \in V_+^L$ is said to be a *completely additive measure* on L if, for any $\bigoplus$-orthogonal system $\{a_i : i \in I\}$, for which the $\bigoplus$-sum $\bigoplus_{i\in I} a_i$ exists in L, we have for any finite subset F of I

$$\Big|\mu\Big(\bigoplus_{i\in I} a_i\Big) - \sum_{i\in F} \mu(a_i)\Big| \le b_F\,, \tag{5.7}$$

where $\{b_F\} \downarrow 0$ and $b_{F_1} \le b_{F_2}$ whenever $F_2 \subseteq F_1$. Due to (4.4), (5.7) is defined correctly, and we shall write $\mu\Big(\bigoplus_{i\in I} a_i\Big) = \sum_{i\in I} \mu(a_i)$.

If the index set I in (5.7) is only countable, we say that μ is *σ-additive*, and we write $\mu\Big(\bigoplus_{i=1}^{\infty} a_i\Big) = \sum_{i=1}^{\infty} \mu(a_i)$.

Since any Dedekind complete Riesz space is *Archimedean*, i.e., if, for some $x, y \in V$ with $nx \le y$ for every integer n, we have $x \le 0$, we conclude that $\mu(0) = 0$. Indeed, for any finite subset F of I with $\Big|\mu\Big(\bigoplus_{i\in I} a_i\Big) - \sum_{i\in F} \mu(a_i)\Big| \le b_F$, where $a_i = 0$ for any $i \in I$, we have $(\mathrm{card}\, F - 1)\,|\,\mu(0)| \le b_F \downarrow 0$, so that $\mu(0) = 0$.

Moreover, any completely additive measure is σ-additive, and any σ-additive measure is finitely additive.

We denote by $a(L,V)_+$, $\sigma a(L,V)_+$, and $ca(L,V)_+$ the sets of all positive finitely additive, σ-additive, and completely additive measures, respectively, from V_+^L.

It is not hard to prove that a positive additive measure μ on L is σ-additive, or completely additive, iff

$$\Big\{\sum_{i=1}^{n} \mu(a_i)\Big\} \uparrow \mu\Big(\bigoplus_{i=1}^{\infty} a_i\Big), \tag{5.8}$$

or

$$\Big\{\sum_{i\in F} \mu(a_i)\Big\}_F \uparrow \mu\Big(\bigoplus_{i\in I} a_i\Big), \tag{5.9}$$

where F runs over all finite subsets of I, whenever $\bigoplus_{i=1}^{\infty} a_i$, or $\bigoplus_{i\in I} a_i$, respectively, exists in L.

6. Yosida–Hewitt decomposition

In the present section, we prove the main results of the paper – generalization of the Yosida-Hewitt decomposition theorem for Dedekind complete Riesz space-valued measures on orthoalgebras. This result generalizes that one in [3] and [4].

We recall that our decompositions do not yield the uniqueness of that one. Some partial results concerning the uniqueness are presented in [3]. An element $\mu \in a(L,V)_+$ is said to be *weakly purely additive* if

$$\eta \leq_n \mu,\ \eta \in ca(L,V)_+ \implies \eta = 0. \tag{6.1}$$

If (6.1) holds for $\eta \in \sigma a(L,V)_+$, μ is said to be *purely additive*. An element $\mu \in \sigma a(L,V)_+$ is said to be *purely σ-additive*, if (6.1) holds.

THEOREM 6.1. *Every positive finitely additive measure μ on an orthoalgebra L with values in a Dedekind complete Riesz space V can be expressed as a sum*

$$\mu = \xi + \eta,$$

where $\xi \in ca(L,V)_+$, and η is a positive weakly purely additive measure on L.

Proof. First we observe that if μ_1 and μ_2 are elements of $ca(L,V)_+$, then $\mu_1 + \mu_2 \in ca(L,V)_+$, where $(\mu_1+\mu_2)(a) := \mu_1(a) + \mu_2(a)$, $a \in L$. Indeed, this follows from (5.9) and (4.3).

Define $\Gamma_\mu = \{\gamma \in ca(L,V)_+ : \gamma \leq_n \mu\}$. Then Γ_μ is non-empty because it possesses the zero function. Let $\Gamma_0 = \{\gamma_i\}$ be a chain in Γ_μ with respect to the natural ordering $\leq_n$, and define

$$\gamma_0(c) = \bigvee_i \gamma_i(c), \quad c \in L.$$

Since $0 \leq \gamma_i(c) \leq \gamma_i(1) \leq \mu(1)$, and V is Dedekind complete, $\gamma_0(c)$ is defined correctly on L. Moreover, $\gamma_0 \in a(L,V)_+$. Indeed, let $a \oplus b$ be defined in L. Then $\{\gamma_i(a)\}$ and $\{\gamma_i(b)\}$ are equidirected, and $\gamma_i(a) \uparrow \gamma_0(a)$, $\gamma_i(b) \uparrow \gamma_0(b)$. By (4.3) we conclude that $\gamma_i(a \oplus b) = ((\gamma_i(a) + \gamma_i(b)) \uparrow (\gamma_0(a) + \gamma_0(b))$. Since $\gamma_i(a \oplus b) \uparrow \gamma_0(a \oplus b)$, we obtain $\gamma_0(a \oplus b) = \gamma_0(a) + \gamma_0(b)$.

From the definition of γ_0 we conclude that $\{\gamma_0(c) - \gamma_i(c)\} \downarrow 0$ for any $c \in L$. Now let $c \in L$ be arbitrary. Due to inequalities

$$\begin{aligned} 0 &\leq \gamma_0(c) - \gamma_i(c) = \gamma_0(1)\gamma_i(1) - (\gamma_0(c^\perp) - \gamma_i(c^\perp)) \leq \\ &\leq \gamma_0(1) - \gamma_i(1) \downarrow 0, \end{aligned}$$

we conclude that $\{\gamma_0(c) - \gamma_i(c)\} \downarrow 0$ uniformly for any $c \in L$.

We claim to show that $\gamma_0 \in ca(L,V)_+$. Let $a = \bigoplus_{j\in I} a_j$ exists in L. Then, for any finite subsets F of I, have

$$\begin{aligned} 0 &\leq \gamma_0(a) - \sum_{j\in F} \gamma_0(a_j) = \gamma_0\Big(a \ominus (\bigoplus_{j\in F} a_j)\Big) = \\ &= \Big(\gamma_0\Big(a \ominus (\bigoplus_{j\in F} a_j)\Big) - \gamma_i\Big(a \ominus (\bigoplus_{j\in F} a_j)\Big)\Big) + \gamma_i\Big(a \ominus (\bigoplus_{j\in F} a_j)\Big) \leq \\ &\leq p_i + b_F^i, \end{aligned}$$

where $\{p_i\} \downarrow 0$, $\{b^i_F\}_F \downarrow 0$ for every i, and F is any finite subset of I. Then

$$0 \le \gamma_0(a) - \bigvee_F \sum_{j\in F} \gamma_0(a_j) \le p_i \downarrow 0,$$

so that $\gamma_0(a) = \sum_{j\in I} \gamma_0(a_j)$.

Therefore, this with $\gamma_0 \le_n \mu$ means that γ_0 is a majorant of Γ_0 in Γ_μ. It follows from Zorn's lemma that Γ_μ contains a maximal element ξ which belongs to $ca(L,V)_+$ and $\xi \le_n \mu$.

Put $\eta = \mu - \xi$, clearly that $\eta \in a(L,V)_+$. To finish the proof, we show that η is weakly purely additive. Let $\gamma \in ca(L,V)_+$ be such that $\gamma \le_n \eta = \mu - \xi$, so that $\gamma + \xi \le_n \mu$. Because $\gamma + \xi \in ca(L,V)_+$, the maximality of ξ in Γ_μ implies $\gamma = 0$. □

THEOREM 6.2. *Every positive finitely additive measure μ on an orthoalgebra L with values in a Dedekind complete Riesz space V can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in \sigma a(L,V)_+$, and η is a positive purely additive measure on L.*

Proof. It follows the same ideas as the proof of Theorem 6.1, it suffices to change $ca(L,V)_+$ to $\sigma a(L,V)_+$. □

THEOREM 6.3. *Every positive σ-additive measure μ on an orthoalgebra L with values in a Dedekind complete Riesz space V can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in ca(L,V)_+$, and η is purely σ-additive.*

Proof. It is identical with that in Theorem 6.1 changing $a(L,V)_+$ to $\sigma a(L,V)_+$. □

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