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ON THE STRUCTURE OF T_s–TRIBES

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ABSTRACT. We extend the results of Navara [8] for tribes of fuzzy sets closed under product to the case of a general strict t-norm. We show that any $\boldsymbol{T}_s$-tribe, $s \in (0,\infty)$, is a weakly generated tribe and, consequently, a $\boldsymbol{T}$-tribe for any measurable t-norm $\boldsymbol{T}$.

1. Preliminaries

Let $\boldsymbol{X}$ be a non-empty set. A function $\boldsymbol{f}\colon \boldsymbol{X} \to [0,1]$ has been called a *fuzzy subset* of $\boldsymbol{X}$ (Zadeh [11]). This generalizes the concept of a (Cantorian) subset A of $\boldsymbol{X}$ which can be identified with its characteristic function $1_A\colon \boldsymbol{X} \to \{0,1\}$ defined by $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$. 1_A is called a *crisp subset* of $\boldsymbol{X}$. If $\boldsymbol{f}$ is a fuzzy subset of $\boldsymbol{X}$, then $\boldsymbol{f}(x)$ is interpreted as the degree of membership of the point x in $\boldsymbol{f}$. The collection of all fuzzy subsets of $\boldsymbol{X}$ is denoted by $\mathscr{F}(\boldsymbol{X})$, $\mathscr{F}(\boldsymbol{X}) = [0,1]^{\boldsymbol{X}}$. For a constant $d \in [0,1]$, we denote by $\boldsymbol{d}$ the fuzzy subset $\boldsymbol{d}(x) = d$ for all $x \in \boldsymbol{X}$.

The extension of set-theoretic operations of intersection, union and complementation to the fuzzy set operations is done pointwise,

(i) $(\boldsymbol{f} \cap \boldsymbol{g})(x) = \boldsymbol{T}(\boldsymbol{f}(x),\, \boldsymbol{g}(x))$,

(ii) $(\boldsymbol{f} \cup \boldsymbol{g})(x) = \boldsymbol{S}(\boldsymbol{f}(x),\, \boldsymbol{g}(x))$,

(iii) $\boldsymbol{f}^c(x) = \boldsymbol{C}(\boldsymbol{f}(x))$,

for any $x \in \boldsymbol{X}$, $\boldsymbol{f}, \boldsymbol{g} \in \mathscr{F}(\boldsymbol{X})$. Here $\boldsymbol{C}$ is a complementation operator, $\boldsymbol{T}$ is a t-norm, and $\boldsymbol{S}$ is its $\boldsymbol{C}$-dual t-conorm. Hence

$$\boldsymbol{C}(a) = g^{-1}(1 - g(a)), \qquad a \in [0,1].$$

For more details see [6] or [9].

The concept of t-norm based tribes (on $\boldsymbol{X}$) is a natural generalization of the notion of a σ-algebra of subsets (of $\boldsymbol{X}$).

DEFINITION 1.1. Let $\boldsymbol{T}$ be a t-norm. A system $\mathscr{T} \subset \mathscr{F} \equiv \mathcal{F}(\boldsymbol{X})$ is called a $\boldsymbol{T}$-*tribe* if

(i) $\boldsymbol{0} \in \mathscr{T}$,

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(ii) $f \in \mathscr{T}$ implies $f' = 1 - f \in \mathscr{T}$,
(iii) $\{f_n\}_{n\in N} \subset \mathscr{T}$ implies $T_{n\in N} f_n \in \mathscr{T}$.

For more details see [1, 2, 6]. Note that a T-tribe is closed also under countable unions induced by a dual t-conorm S.

DEFINITION 1.2. Let $\mathscr{S} \subset 2^X$ be a σ-algebra of crisp subsets of X. The system $\mathscr{T}(\mathscr{S})$ of all $\mathscr{S}$-measurable fuzzy subsets of X is called a *generated tribe.*

It is easy to see that a generated tribe is a T-tribe for any measurable t-norm T. However, the converse is not true, in general. The aim of this paper is to study the structure of T-tribes for some special types of t-norms.

DEFINITION 1.3. A t-norm T is called a *strict t-norm* if it is continuous and strictly increasing on $(0,1) \times (0,1)$.

Note that the continuity of a t-norm T implies its measurability (with respect to Borel sets) and that it is a natural requirement adopted in almost all applications. Further, a continuous t-norm T is said to be *Archimedean* if $T(a,a) < a$ for all $a \in (0,1)$. Any continuous Archimedean t-norm T is generated by an additive generator $f\colon [0,1] \to [0,\infty]$, f is continuous, strictly decreasing and $f(1) = 0$, [5], i.e.,

$$T(a,b) = f^{-1}\Big(\min\big(f(0),\, f(a) + f(b)\big)\Big), \qquad a, b \in [0,1]. \tag{1}$$

A t-norm is strict if and only if it has an unbounded additive generator f, $f(0) = \infty$. In this case, $T(a,b) = f^{-1}\big(f(a) + f(b)\big)$. If $f(0)$ is finite, then T generated via (1) is called a *nilpotent t-norm.* Note that the generators are determined by the corresponding t-norm uniquely up to a positive multiplicative constant. For the dual t-conorms S, several properties of T (continuity, strictness, nilpotency) are preserved. More details can be found in [1, 2, 5, 6, 9].

2. T_s-tribes

One of the most important families of t-norms is the Frank family [3] of fundamental t-norms $\{T_s; s \in [0,\infty]\}$, where

$$T_s(a,b) = \begin{cases} \min(a,b), & \text{for } s = 0,\\ a \cdot b, & \text{for } s = 1,\\ \max(0, a+b-1), & \text{for } s = \infty,\\ \log_s\big(1 + (s^a = 1)\cdot(s^b - 1)/(s-1)\big), & \text{otherwise}. \end{cases}$$

Their corresponding t-conorms are

$$S_s(a,b) = \begin{cases} \max(a,b), & \text{for } s = 0,\\ a + b - a\cdot b, & \text{for } s = 1,\\ \min(a+b, 1), & \text{for } s = \infty,\\ 1 - \log_s\big(1 + (s^{1-a} - 1)\cdot(s^{1-b} - 1)/(s-1)\big), & \text{otherwise}. \end{cases}$$

Recall that the pairs (T_s, S_s), $s \in (0,\infty]$, are the only continuous Archimedean solutions of the functional equation (see [3])

$$T(a,b) + S(a,b) = a + b \quad \text{for any} \quad a,b \in [0,1].$$

Further, T_s is a strict t-norm for any $s \in (0,\infty)$. T_∞ is a nilpotent t-norm and T_0 is the maximal t-norm, i.e. for any t-norm T and any $a,b \in [0,1]$ it holds $T(a,b) \le T_0(a,b)$.

We recall some results of Klement and Butnariu [1, 2, 4] for T_s-tribes.

THEOREM 2.1. *For any fundamental t-norm T_s with $s > 0$, and for each T_s-tribe $\mathscr{T}$ on X we have $\mathscr{T} \subset \mathscr{F}(\mathscr{S})$, where $\mathscr{S}$ is the σ-algebra of all crisp subsets of X contained in $\mathscr{T}$.*

THEOREM 2.2. *For any fundamental t-norm T_s with $s > 0$, a T_s-tribe $\mathscr{T}$ on X is a generated tribe if and only if $\mathscr{T}$ contains all the constant fuzzy subsets of X.*

THEOREM 2.3. *For any fundamental t-norm T_s with $s \in (0,\infty)$, each T_s-tribe on X is a T_∞-tribe on X, too. Further, each T_∞-tribe on X is also a T_0-tribe on X.*

In [7], we have shown the following property of T_s-tribes.

THEOREM 2.4. *Let X be countable. Then for any fundamental t-norm T_s with $s \in (0,\infty)$, each T_s-tribe $\mathscr{T}$ on X is a semigenerated tribe on X, i.e., there is a crisp partition (Y,Z) of X such that $\mathscr{T}|Y$ is a generated tribe on Y and $\mathscr{T}|Z$ is a σ-algebra of crisp subsets of Z.*

It is obvious that a semigenerated tribe $\mathscr{T}$ is a T-tribe for any (measurable) t-norm T. Our question about the structure of T_s-tribes, $s \in (0,\infty)$, on a general universe X (c.f. see [6, 7]) was recently solved by Navara [8] in the case $s = 1$, i.e., when T_s is the usual product on the unit interval $[0,1]$.

THEOREM 2.5. *Each T_1-tribe $\mathscr{T}$ on X is a weakly generated tribe on X, i.e., there is a σ-ideal Δ in $\mathscr{S}$ (crisp subsets from $\mathscr{T}$) such that*

$$\mathscr{T} = \{ f \in \mathscr{T}(\mathscr{S});\ Df \in \Delta \},$$

where $Df = \{ x \in X;\ f(x) \in (0,1) \}$.

We can show again that a weakly generated tribe is a T-tribe for all (measurable) t-norms T. It is easy to see that a reverse assertion is also true because of the measurability of the t-norm T_1, i.e., a system $\mathscr{T}$ of fuzzy subsets of X is a T-tribe for all (measurable) t-norms T if and only if $\mathscr{T}$ is a weakly generated tribe. Note that any generated tribe is a semigenerated tribe with $Y = X$ and $Z = \emptyset$. Further, any semigenerated tribe is a weakly generated tribe with $\Delta = \mathscr{S}|Y$.

If $\mathscr{T}$ is a T_∞-tribe on X, then it need not be a weakly generated tribe. Take, e.g., $X = \{x\}$ (a singleton) and $\mathscr{T}_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$, $n = 2,3,\dots$. A similar example can be introduced for any nilpotent t-norm T. This reduces our attention to the strict t-norms only. For fundamental t-norms T_s this means that $s \in (0,\infty)$.

3. Characterization of T_s-tribes, $s \in (0, \infty)$

Let T be a given strict t-norm and let S be its dual t-conorm. Let $\mathcal{T}$ be a T-tribe on X. For each $k, n \in \mathbb{N}$ we define a non-decreasing function on $[0,1]$

$$T^{k,n}(t) = T^n\big(S^k(t)\big),$$

where $T^1(t) = S^1(t) = t$, $T^{n+1}(t) = T\big(T^n(t), t\big)$, $S^{n+1}(t) = S\big(S^n(t), t\big)$, $n = 1, 2, \dots, t \in [0,1]$.
It is easy to see that $\mathcal{T}$ is closed under $T^{k,n}$ for any $k, n \in \mathbb{N}$, i.e., if $f \in \mathcal{T}$, then $T^{k,n} \circ f \in \mathcal{T}$, too.

Lemma 3.1. *For each $a, b, t \in (0,1)$, $a < b$, there are $k, n \in \mathbb{N}$ such that $T^{k,n}(t) \in [a,b]$.*

Proof. Let f be an additive generator of T. Then $f(0) = \infty$ and g, $g(t) = f(1-t)$, is an additive generator of S. We have

$$T^n(t) = f^{-1}\big(n \cdot f(t)\big) \quad \text{and} \quad S^n(t) = g^{-1}\big(n \cdot g(t)\big) \quad \text{for any} \quad t \in (0,1), \; n \in \mathbb{N}.$$

Hence, for any $t \in (0,1)$, k, $n \in \mathbb{N}$, it is

$$T^{k,n}(t) = f^{-1}\Big(n \cdot f\big(g^{-1}(k \cdot g(n))\big)\Big) = f^{-1}\Big(n \cdot f\Big(1 - f^{-1}\big(k \cdot \big(f(1-t)\big)\big)\Big)\Big).$$

Let $a, b, t \in (0,1)$, $a < b$, be given. Put $q = f(a) - f(b) \in (0, \infty)$. Then there is $k \in \mathbb{N}$ such that

$$k \geq f\big(1 - f^{-1}(q)\big) / f(1-t).$$

It follows that

$$f(a) - f(b) = q \geq f\Big(1 - f^{-1}\big(k \cdot \big(f(1-t)\big)\big)\Big) > 0,$$

which ensures the existence of $n \in \mathbb{N}$ such that

$$\infty > f(a) \geq n \cdot f\Big(1 - f^{-1}\big(k \cdot \big(f(1-t)\big)\big)\Big) \geq f(b) > 0.$$

The last inequalities are equivalent to

$$0 < a \leq T^{k,n}(t) \leq b < 1.$$

□

Proposition 3.2. *Let $\mathcal{T}$ be both a T_0-tribe and a T-tribe, with T a strict t-norm. Let $f \in \mathcal{T}$. Then $\mathcal{T}|Df$ contains all constant fuzzy subsets of Df.*

Proof. Put

$$U(t) = \sup\big\{\inf(T^{k,n}(t),\, 1 - T^{k,n}(t)\};\; k,\; n \in \mathbb{N}\big\}$$

for $t \in [0,1]$. $\mathscr{T}$ is a T-tribe and hence it is closed under both $T^{k,n}$ and $(1-T^{k,n})$. Further, $\mathscr{T}$ is also a T_0-tribe and hence it is closed under infima and suprema of countable classes of elements of $\mathscr{T}$. Consequently, $\mathscr{T}$ is closed under U, i.e., for each $f \in \mathscr{T}$ also $U \circ f \in \mathscr{T}$ (here $(U \circ f)(x) = U(f(x))$, $x \in X$). It is easy to see that $U(t) \le \frac{1}{2}$ for any $t \in (0,1)$ and that $U(0) = U(1) = 0$. Due to Lemma 3.1, for arbitrary small $\varepsilon > 0$ and any $t \in (0,1)$ there are some $k,n \in \mathbb{N}$ so that $T^{k,n}(t) \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2}\right]$. It follows that $U(t) = \frac{1}{2}$ for all $t \in (0,1)$.

Take a fuzzy subset $f \in \mathscr{T}$. Then $U \circ f \in \mathscr{T}$ and $U \circ f(x) = \frac{1}{2}$ for any $x \in Df$. Following Lemma 1 of [7], $\mathscr{T}|Df$ contains all constant fuzzy subsets of Df. □

THEOREM 3.3. *Let $\mathscr{T}$ be a T_s-tribe for some $s \in (0,\infty)$. Then $\mathscr{T}$ is a weakly generated tribe.*

Proof. Let $\mathscr{T}$ be a T_s-tribe, $s \in (0,\infty)$. Then $\mathscr{T}$ is also a T_∞-tribe and consequently it is also a T_0-tribe, see Theorem 2.3. Take a fuzzy subset f contained in $\mathscr{T}$. By Proposition 3.2, $\mathscr{T}|Df$ contains all the constant fuzzy subsets of Df. It is evident that $\mathscr{T}|Df$ is a T_s-tribe on Df. By Theorem 2.2, $\mathscr{T}|Df$ is a generated tribe on Df, $\mathscr{T}|Df = \mathscr{F}(\mathscr{S}|Df)$.

Let f, $g \in \mathscr{T}$. Let T be any measurable t-norm. Put

$$h = \sup\{\inf(f,f'),\ \inf(g,g')\}.$$

Then $h \in \mathscr{T}$ and $\mathscr{T}|Dh$ is a generated tribe on Dh, i.e., $\mathscr{T}|Dh$ is closed under T. Hence $T(f|Dh, g|Dh) = T(f,g)|Dh \in \mathscr{T}|Dh$, i.e., there is some fuzzy subset p of X, $p \in \mathscr{T}$, such that $p|Dh = T(f,g)|Dh$.

Put

$$r = \sup_{n\in N}\{(S_n)^n \circ U \circ h\} = \begin{cases} 0, & \text{if } f(x) \text{ and } g(x) \in \{0,1\},\\ 1, & \text{otherwise,}\end{cases}$$

$$f^* = \inf_{n\in N}\{(T_s)^n \circ f\} = \begin{cases} 1, & \text{if } f(x) = 1,\\ 0, & \text{otherwise,}\end{cases}$$

$$g^* = \inf_{n\in N}\{(T_s)^n \circ g\} = \begin{cases} 1, & \text{if } g(x) = 1,\\ 0, & \text{otherwise.}\end{cases}$$

Then r, f^* and g^* are contained in $\mathscr{T}$ and

$$T(f,g) = \sup\{T_s(r,p),\ T_s(f^*,g^*)\} \in \mathscr{T}.$$

We have just shown that $\mathscr{T}$ is closed under arbitrary measurable t-norm T. $\mathscr{T}$ is also closed under countable suprema and infima as it is also a T_0-tribe and consequently $\mathscr{T}$ is a T-tribe for any measurable t-norm T. Especially, $\mathscr{T}$ is also a T_1-tribe on X and, by Theorem 2.3 of Navara [8], $\mathscr{T}$ is a weakly generated tribe. □

COROLLARY 3.4. *The following three assertions are equivalent:*

(i) $\mathscr{T}$ *is a weakly generated tribe on* X.

(ii) $\mathscr{T}$ *is a* T*-tribe on* X *for all measurable t-norms* T.

(iii) *For some fundamental t-norm* T_s *with* $s \in (0, \infty)$, $\mathscr{T}$ *is a* T_s*-tribe on* X.

We have shown that if $\mathscr{T}$ is a T-tribe on X for some strict fundamental t-norm T (i.e., $T = T_s$ for some $s \in (0, \infty)$), then $\mathscr{T}$ is closed under any measurable t-norm T (and hence under any measurable t-conorms S). Is this true for any strict t-norm T?

OPEN PROBLEM 3.5. *Let* $\mathscr{T}$ *be a* T*-tribe on* X *for some strict t-norm* T. *Does this requirement ensure the closedness of* $\mathscr{T}$ *with respect to any measurable t-norm and t-conorm?*

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