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# CLOSURE OPERATOR INDUCED BY ORTHOGONALITY AND ORTHOMODULAR LAW

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ABSTRACT. Orthogonality relation is the basic notion for the construction of two lattices of all closed sets and of all open sets, respectively. If these lattices are orthomodular then for any set contained in the intersection of these lattices, orthocomplementation in the lattice of closed sets and orthocomplementation in the lattice of open sets are equal. This condition is not sufficient.

Let $A^c$ denote the set-theoretic complement of $A$.

System with orthogonality is a couple $(\mathbf{V}, \perp)$ such that $\mathbf{V}$ is a nonempty set and the orthogonality relation $\perp$ fulfills irreflexivity and symmetry conditions.

$$(\forall x \in \mathbf{V} \;\; \neg (x \perp x) \text{ — irreflexivity},$$

$$x \perp y \Leftrightarrow y \perp x \text{ — symmetry}).$$

The orthogonality relation induces an operator which maps every subset $A \subseteq \mathbf{V}$ to its polar defined by:

$$A^{\perp} = \{x \in \mathbf{V};\; \forall y \in A \;\; x \perp y\}.$$

We will use the following properties of polar operator:

$$A \subseteq B \Rightarrow B^{\perp} \subseteq A^{\perp},$$
$$\emptyset^{\perp} = \mathbf{V} \;;\; \mathbf{V}^{\perp} = \emptyset,$$
$$(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp},$$
$$A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp},$$
$$A^{\perp} \subseteq A^c.$$

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The second iteration of this polar operator is a closure operator in the sense of universal algebra, i.e.

$$\emptyset^{\perp\perp} = \emptyset\,;$$
$$A \subseteq B \Rightarrow A^{\perp\perp} \subseteq B^{\perp\perp}\,;$$
$$A \subseteq A^{\perp\perp}\,;$$
$$A^{\perp\perp\perp\perp} = A^{\perp\perp}\,;$$
$$A^{\perp\perp\perp} = A^{\perp}\,.$$

It is well-known [1] that the system of all closed subsets of $\mathbf{V}$ forms a complete ortholattice. The meet of a family of elements of this lattice is the set-theoretic intersection and the join is the closure of the set-theoretic union. The orthocomplementation is equal to the polar operator. The closure operator induces an interior operator defined by: $A \to A^{c\perp\perp c}$, and the lattice of all open sets and the lattice of all closed sets are isomorphic. We note that an open set is defined by $A = A^{c\perp\perp c}$. Orthocomplementation in "open" lattice is defined by: $A \to A^{c\perp c}$. The join in "open" lattice is the set-theoretic union and the meet is defined as the interior of the set-theoretic intersection.

We will call *clopen set* any set which is closed and open simultaneously.

**EXAMPLE 1.** Let $\mathbf{V} = \{a,b,c,d\}$ and $(a,b)$, $(b,c)$, $(c,d)$ is the list of all orthogonal couples. $\{a,c\}$ is a clopen set. Its polar is $\{b\}$, and $\{a,c\}^{c\perp c} = \{a,b,d\}$. This orthogonality relation induces the ortholattice of all closed subsets which is not orthomodular. There is a set with different orthocomplements in "closed" lattice and in "open" lattice.

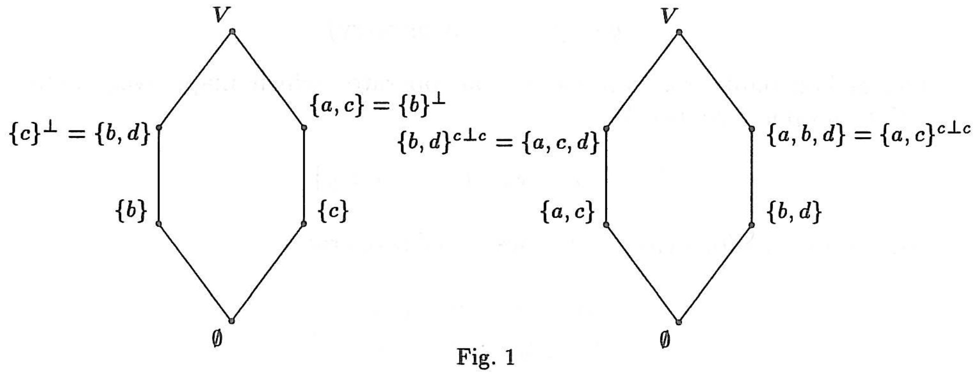


Fig. 1

One can pose the question about generalization of this conjecture. Result of this generalization is Theorem 1. We need the following lemma for simplification of the proof of Theorem 1.

A set is called *orthoregular* if its polar is equal to its set-theoretic complement.

**LEMMA 1.** *A set $A$ is orthoregular iff it is clopen and its orthocomplements with respect to closure and interior are equal, i.e.*

$$A^{\perp} = A^{c} \iff A^{\perp\perp} = A = A^{c\perp\perp c} \quad \textit{and} \quad A^{\perp} = A^{c\perp c}\,.$$

P r o o f. a.) Orthoregularity is assumed: $A^{\perp} = A^{c}$ implies $A^{\perp\perp} \subseteq A^{c\perp} \subseteq A^{cc} = A$. If the closure is a subset of considered set then it is closed. $A = A^{cc} = A^{\perp c} = A^{\perp\perp\perp c} = A^{c\perp\perp c}$. $A$ is open. $A^{c\perp c} = A^{\perp\perp c} = A^{c} = A^{\perp}$.

b.) $A$ is clopen and orthocomplementations are equal: $A^{c} = A^{\perp\perp c} \subseteq A^{c\perp c} = A^{\perp}$. The reverse inclusion is clear. □

**Theorem 1.** ([4]). *If the lattice of all closed subsets is orthomodular then any clopen subset is orthoregular.*

P r o o f. Let $A$ be a set which is clopen and not orthoregular. $A^{\perp}$ and $A^{c\perp\perp}$ are closed and $A^{\perp} \subseteq A^{c\perp\perp}$. If the orthomodular law holds then $A^{c\perp\perp} = [A^{\perp} \cup (A^{c\perp\perp} \cap A^{\perp\perp})]^{\perp\perp}$.

If $A$ is clopen then $A^{c\perp\perp} = A^{c}$ and $A^{\perp\perp} = A$ implies $A^{c\perp\perp} \cap A^{\perp\perp} = \emptyset$ and $A^{c} \subseteq A^{c\perp\perp} = A^{\perp}$. This implies orthoregularity and it is a contradiction. □

**Example 2.** Let $\mathbf{V} = \{a, b, c, d, e\}$ with the list of all orthogonal couples: $(a, b)$, $(b, c)$, $(c, d)$, $(d, e)$, $(e, a)$. If cardinality of $A \subseteq \mathbf{V}$ is greater than 2, then $A^{\perp} = \emptyset$. This implies that $\mathbf{V}$ is a unique closed subset with cardinality greater than 2. Similarly $\emptyset$ is a unique open subset with cardinality less than 3. The list of all clopen subsets contains only $\mathbf{V}$ and $\emptyset$. Orthoregularity of these sets is clear. If we verify orthomodular law for a couple of closed sets $\{a\}$ and $\{a, c\}$ we obtain negative result. The lattice of all closed subsets is not orthomodular.

The condition from Theorem 1 is not a characterization of the orthomodularity.

The set of orthogonal couples consisting of three couples from Example 1 is the special tool for formulation of some sufficient conditions.

Let $(\mathbf{V}, \perp)$ be a set with orthogonality. $\{a, b, c, d\} \subseteq \mathbf{V}$ is called a *critical quadrangle* if the following list of conditions holds:

$$a \perp b,\ b \perp c,\ c \perp d,\ \neg(a \perp c),\ \neg(b \perp d),\ \neg(a \perp d).$$

**Theorem 2.** *If a system with orthogonality* $(\mathbf{V}, \perp)$ *contains no critical quadrangle, then the lattice of all closed subsets is orthomodular.*

P r o o f. We assume that orthomodular law does not hold: There are $A, B$ such that $A = A^{\perp\perp}$, $B = B^{\perp\perp}$, $A \subsetneq B$ and $[A \cup (B \cap A^{\perp})]^{\perp\perp} \subsetneq B$. The last inclusion is equivalent to $B^{\perp} \subsetneq [A \cup (B \cap A^{\perp})]^{\perp}$. This fact implies that $\exists\, z \in A^{\perp} \cap (B \cap A^{\perp})^{\perp}$ such that $z \notin B^{\perp}$. It is easy to see that $z \notin B$. It follows from $z \notin B^{\perp}$ that there is $w \in B$, $w \notin A$ and $\neg\,(z \perp w)$. We can choose $u \in A$ and $v \in B^{\perp}$ such that $\neg\,(u \perp w)$ and $\neg\,(v \perp z)$. Then $\{z, u, v, w\}$ forms a critical quadrangle. □

The condition from Theorem 2 is not necessary.

**Example 3.** $\mathbf{V} = \{a, b, c, d, e\}$ and the list of all orthogonal couples is: $(a, b)$, $(b, c)$, $(c, a)$, $(a, d)$, $(b, e)$. So $\{d, a, b, e\}$ is a critical quadrangle. The lattice of all closed subsets is isomorphic to Boolean algebra with 3 atoms and this lattice is orthomodular.

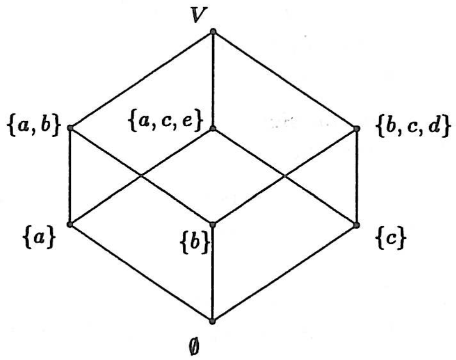


Fig. 2

In [2], [3], the notion of quasimanual of experiments is considered. It is a generalization of the orthoalgebra and the orthoalgebra is a generalization of orthomodular poset. There is a procedure of construction of *partition quasimanual* (see [2]) from the orthoalgebra. The following example is the simplest example of a difference between the ortholattice and the orthoalgebra.

We define a "new" orthogonality relation in the lattice of all closed subsets of the system with the "old" orthogonality relation:

$$A, B \quad \text{are } \textit{orthogonal} \quad \Longleftrightarrow \quad A \subseteq B^{\perp}.$$

**EXAMPLE 4.** Let $\mathbf{V}$ and the list of orthogonal couples be the same as in Example 1. The lattice of all closed subsets is the smallest nonorthomodular ortholattice. If we take a partial operation ",,joint of orthogonal couple" and we test the axioms of orthoalgebra, we conclude that there are elements $\{b\}$, $\{c\}$, $\{a,c\}$ such that $\{b\}\perp\{c\}$, $\{b\}\perp\{a,c\}$ and $\{b\} \uplus \{c\} = \mathbf{V}$ and $\{b\} \uplus \{a,c\} = \mathbf{V}$. This is a contradiction of unicity of orthocomplement which is assumed in axiomatic system of orthoalgebra. This implies that an ortholattice with the operation described in this example is not an orthoalgebra.

The notion of an ortholattice is not the same as an orthoalgebra but the procedure of construction of partition quasimanual is usable for ortholattices. A *partition of unity* in ortholattice is an indexed system of mutually orthogonal elements such that there is unique upper bound of this system – the greatest element. Two different partitions of unity are noncomparable in the sense of set-theoretic inclusion. The system of all (or all finite, or all countable) partitions of unity forms a quasimanual (noncomparability is sufficient condition). A partition of unity is a test and every subset of a test is an *event.* An event $A$ is *orthogonal* with an event $B$ if $A \cup B$ is an event and $A$ and $B$ are disjoint. $A$, $B$ are *operational complements* if they are orthogonal and $A \cup B$ is a test. $A$, $B$ are *operationally perspective* if they have a common operational complement. The

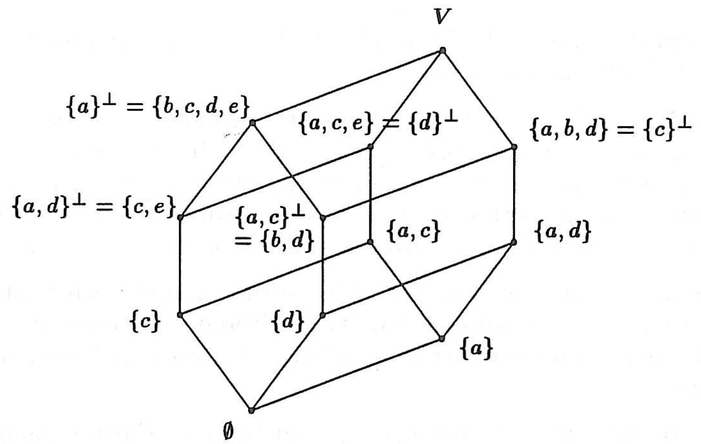


Fig. 3

transitive closure of a relation of operationally perspectivity is the equivalence relation and it allowes to consider a factor structure. This factor structure is a poset ordered by composition of set – theoretical inclusion and equivalence relation described above. We omit the detailed construction (see [2]). If we choose two events such that they form a complemented couple, then two different cases can be considered. Complementarity is compatible with transitive closure of perspectivity (a quasimanual is a manual, see [2]) or the opposite case. In [4] a finite ortholattice is investigated and special partitions of unity are considered: $\{x\}^{\perp\perp}$ are elements of partition of unity.

**THEOREM 3.** ([4]). *Let* $(\mathbb{L}, \vee, \wedge, \perp, 0, 1)$ *be a finite ortholattice. Let* $(\mathbb{V}, \perp) = (\mathbb{L} - \{0,1\}, \perp)$ *be a system with orthogonality defined for ortholattices above. Then orthomodularity of the lattice of all closed subsets implies the fact that the partition quasimanual induced above is a manual.*

P r o o f . Let the partition quasimanual do not be a manual. Then there are events $A, B, C, D$ such that couples $(A,B)$, $(B,C)$, $(C,D)$ are operational complements and $(A,D)$ is not a couple of operational complements. $C \subseteq D^\perp$ and $A, C$ are operationally perspective. If we assume the orthomodular law, we can obtain by computing $A^{\perp\perp} = B^\perp = C^{\perp\perp} = D^\perp$. We use the orthomodular law for $A^{\perp\perp} \subseteq D^\perp$ and we obtain $A^{\perp\perp} = D^\perp$. This follows from the fact that $D^\perp = B^\perp$ (a conclusion of orthomodular law). This implies that $A$, $D$ are operational complements which is a contradiction. □

The following question is natural: Is there a generalization of this result on some greater class of ortholattices? A generalization on the class of ortholattices with tests of partition quasimanual having finite cardinality is without problem because the proof of Theorem 3 does not use the finiteness of the lattice. If we assume countable or total complete operations on the starting lattice, we can enlarge the partition quasimanual.

The second question is: Is the result of Theorem 3 a characterization of orthomodularity? The answer is negative.

**EXAMPLE 5.** Let $\mathbf{V} = \{a,b,c,d,e\}$ and the list of orthogonality couples is: $(a,b)$, $(a,c)$, $(a,d)$, $(a,e)$, $(b,c)$, $(c,d)$, $(d,e)$. In this example there is no configuration of atomic partitions of unity such that condition "to be a manual" is in contradiction. The ortholattice of all closed subsets is reducible and it is a product of two-element Boolean algebra and the ortholattice from Example 4.

There is an example of a system with orthogonality such that the quasi-manual of atomic decompositions of unity is without "bad configuration" in the sense of Theorem 3 and the lattice of all closed subsets is irreducible and not orthomodular.

**EXAMPLE 6.** Let $\mathbf{V} = \{a,b,c,d,e,f\}$ and the list of orthogonality couples is: $(a,b)$, $(a,c)$, $(b,c)$, $(b,d)$, $(b,e)$, $(c,e)$, $(c,f)$, $(d,e)$, $(e,f)$. Irreducibility is tested by "nonorthogonality relation" which graph is continuous (for more precise analysis see [5]).

In this paper a collection of criteria for orthomodularity of lattice of all closed subsets induced from orthogonality relation is considered. The question of sufficiency of this collection is open.

## REFERENCES


[1] BIRKHOFF, G.: *Lattice Theory*, Providence R.I, 1967.

[2] GUDDER, S. P.: *Quantum Probability*, Academic Press, Inc., 1988.

[3] FOULIS—GREECHIE—RÜTTIMANN: *Supports in Test Spaces*, Penultimate version.

[4] KONÔPKA, P.: *Representation of complete ortholattices by sets with orthogonality*, Internat. J. Theoret. Phys. **32** (1993) (to appear).

[5] ZAPATRIN, R. R.: *Graphs, polarities and finite ortholattices*, Proceedings of the Second Winter School on Measure Theory Liptovský Ján, January 7–12, 1990 Czechoslovakia, 213–218.


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