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# CENTRAL ENVELOPES OF ORTHOMODULAR LATTICES

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*Dedicated to the memory of Tibor Neubrunn*

ABSTRACT. Suppose that $\{L_\alpha \mid \alpha \in I\}$ is a collection of orthomodular lattices and suppose that $B$ is a Boolean algebra. An orthomodular lattice $L$ is called a $B$-envelope of $\{L_\alpha \mid \alpha \in I\}$ if $L$ contains every $L_\alpha$ ($\alpha \in I$) and if the centre of $L$ equals $B$. We show in this note that every collection of orthomodular lattices has a $B$-envelope for any Boolean algebra $B$. We then ask an analogous question for $\sigma$-complete and complete orthomodular lattices. In the latter cases we have been able to find only a partial answer (Th. 2 – 4). The results may find applications in the mathematical foundations of quantum theories.

## Introduction and basic notions

In the logico-algebraic foundation of quantum mechanics one often associates the "event structure" of a quantum experiment with an orthomodular lattice (see e.g. [6], [14] and [16]). The family of all "absolutely compatible" events (i.e. those events that are simultaneously measurable with any other event) then correspond to the centre of the orthomodular lattice in question. The following natural problem of whether one can extend a given event structure (or, more generally, a collection of event structures) to an event structure with an arbitrary set of absolutely compatible elements then transfers in the mathematical setup as follows: Can we enlarge orthomodular lattices to a single orthomodular lattice whose centre has been preassigned? In this note we provide some results that shed light on the latter question.

Let us first recall basic notions which will be used in the sequel (for details, see e.g. [14]).

**DEFINITION 1.** An *orthomodular lattice* (abbr. an OML) is a lattice, $L$, with 0 and 1, which is endowed with an orthocomplementation relation, $': L \to L$,

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such that the following four conditions are fulfilled ( $a, b \in L$ ):

(i) if $a \leq b$, then $b' \leq a'$,
(ii) $(a')' = a$,
(iii) $a \vee a' = 1$,
(iv) if $a \leq b$, then $b = a \vee (b \wedge a')$.

Two elements $a, b \in L$ are called *orthogonal* if $a \leq b'$. If $L$ is closed under the formation of the suprema of countably many mutually orthogonal elements in $L$ (resp. the suprema of arbitrary many mutually orthogonal elements in $L$), then $L$ is called a $\sigma$-complete OML (abbr. $\sigma$-OML) (resp. complete OML) (abbr. c-OML).

Let us agree to denote by $L$ (possibly with indices) an OML. Basic examples of OMLs are Boolean algebras or lattices of projections in von Neumann algebras.

**DEFINITION 2.** A couple $a, b \in L$ is called *compatible* if the elements $(a \wedge (a \wedge b)')$, $(b \wedge (a \wedge b)')$ are orthogonal.

The notion of compatibility models the simultaneous measurability in a quantum experiment. It can be seen easily that the above definition coincides with the usual quantum logic definition (see [14]).

**PROPOSITION 3** (see e.g. [14]). *Put $C(L) = \{a \in L \mid a$ is compatible to any $b \in L\}$ and call $C(L)$ the centre of $L$. Then $C(L)$ is a Boolean sub-OML of $L$ (i.e. $C(L)$ is a Boolean algebra when understood with the operations inherited from $L$). Moreover, if $L$ is a $\sigma$-OML (resp. c-OML), then so is its centre as well.*

In the following definition we determine when an OML is "smaller" than the other.

**DEFINITION 4.** Let $K, L$ be OMLs. We say that $L$ *contains* $K$ if there is an injective mapping $e\colon K \to L$ such that the following conditions are satisfied $(a, b \in K)$:

(i) $e(1) = 1$,
(ii) $e(a') = e(a)'$, and
(iii) $e(a_1 \vee a_2) = e(a_1) \vee e(a_2)$.

If both $K, L$ are $\sigma$-complete (resp. complete), then we say that $L$ *contains* $K$ if the conditions (i) and (ii) above are satisfied and the condition (iii) is fulfilled for countably many $a_i$ $(i \in N)$ (resp. for arbitrarily many $a_i$, $i \in I$).

In view of a potential application in quantum theories, it may be worth observing that if $L$ contains $K$ then the compatibility in $K$ transfers to $L$ (compare with a less satisfying situation in orthomodular posets, see e.g. [12] or [14]).

**OBSERVATION 5.** *If $L$ contains $K$ and if $e\colon K \to L$ is the corresponding embedding then $e(a), e(b)$ are compatible elements in $L$ if and only if $a, b$ are compatible elements in $K$.*

Proof. We shall show the necessity, the sufficiency shows similarly. If $e(a)$, $e(b)$ are compatible in $L$, then the elements $e(a) \wedge (e(a) \wedge e(b))'$, $e(b) \wedge (e(a) \wedge e(b))'$ are orthogonal. Thus, we have $(e(a)\wedge(e(a')\vee e(b')) \leq (e(b)\wedge(e(a')\vee e(b')))'$ and therefore $e(a \wedge (a' \vee b')) \leq e(b \wedge (a' \vee b'))'$. This means that $e(a \wedge (a' \vee b')) \vee e(b \wedge (a' \vee b'))' = e(b \wedge (a' \vee b'))'$ which implies the equality $e((a \wedge (a' \vee b')) \wedge (b \wedge (a' \vee b')')' = e(b \wedge (a' \vee b')')$. Since $e$ is injective, we infer that $(a \wedge (a \wedge b)') \vee (b \wedge (a \wedge b)')' = (b \wedge (a \wedge b)')'$. This equality is equivalent with the inequality $a \wedge (a \wedge b)' \leq (b \wedge (a \wedge b)')'$ which means exactly that $a \wedge (a \wedge b)'$ is orthogonal to $b \wedge (a \wedge b)'$. In other words, $a$, $b$ are compatible in $K$ and the proof is complete. □

In what follows we shall be mainly interested in "enveloping" OMLs.

**DEFINITION 6.** Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of OMLs and let $B$ be a Boolean algebra. We say that $L$ is a $B$-*envelope* of $\{L_\alpha \mid \alpha \in I\}$ if $L$ contains every $L_\alpha$ ($\alpha \in I$) and moreover, $C(L) = B$.

## Results

We will now look for $B$-envelopes of OMLs (resp. $\sigma$-OMLs or c-OMLs). In the proofs we sometimes omit the routine technicalities assuming the reader to be familiar with basic facts and constructions in OMLs (see e.g. [14]). The first result concerns "algebraic" OMLs and brings a full answer to our question. (Prior to the formulation of the next result, let us recall an important class of OMLs. An OML is called *concrete* if it can be represented by a collection of subsets of a set and in this representation the orthocomplementation operation agrees with the set-theoretic complementation and the suprema operation of disjoint elements agrees with the set-theoretic union. All Boolean algebras are obviously concrete OMLs (and there are many others – see, e.g., [5] and [6]), the projection OMLs are usually not concrete.)

**THEOREM 1.** *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of OMLs and let $B$ be a Boolean algebra. Then there is a $B$-envelope of $\{L_\alpha \mid \alpha \in I\}$. If moreover every*

*$L_\alpha$ $(\alpha \in I)$ is concrete then the $B$-envelope of $\{L_\alpha \mid \alpha \in I\}$ can be taken concrete, too.*

Proof. Let $D$ be a four-point Boolean algebra and let $K$ be the horizontal sum of the collection $\{L_\alpha \mid \alpha \in I\} \cup \{D\}$ (see e.g. [9]). Then $K$ is obviously a $\{0,1\}$-envelope of $\{L_\alpha \mid \alpha \in I\}$ (the adding of $D$ guarantees the trivial centre of $K$ in case $\operatorname{card} I = 1$). Let $(\Omega, \tilde{B})$, where $\tilde{B} \subset \exp \Omega$, be a set representation of $B$. Let us now denote by $L$ the set of all functions $f : \Omega \longrightarrow K$ which are determined as follows: The range of $f$ is a finite subset of $K$ and moreover, for any $k \in K$ we require that $f^{-1}(k) \in \tilde{B}$. It can be easily seen that $L$ is an OML with $C(L) = B$. Obviously, $L$ contains every $L_\alpha$ – $K$ contains every $L_\alpha$ and $L$ contains $K$ (every $k \in K$ can be mapped on the constant function $f : \Omega \longrightarrow K$ such that $f(\omega) = k$ for any $\omega \in \Omega$). Thus, $L$ is a $B$-envelope of $\{L_\alpha | \alpha \in I\}$. Moreover, if every $L_\alpha (\alpha \in I)$ is concrete then so is $L$, too. Indeed, $L$ is concrete if and only if the two-valued measures on $L$ distinguish the noncompatible elements of $L$ (see e.g. [5] and [6]). By our construction, it is straightforward to show that "enough" two-valued measures on every $L_\alpha$ $(\alpha \in I)$ guarantees "enough" two-valued measures on $L$. The proof is complete. □

Remark. The latter construction of $L$ – sometimes also called a *Boolean power of OMLs* – has been already used a number of times in OMLs (see e.g. [2], [4], [7], [8], [10], [13], etc.). The nature of this construction is universally algebraic (see [1]).

If we want to extend the latter result to the $\sigma$-complete case, which is often the case used in the quantum axiomatics, we face two additional obstacles. Firstly, a Boolean $\sigma$-algebra does not have to admit a set representation and secondly, a countable collection of countable partitions does not have to admit a countable partition refinement. For the time being, we have not been able to resolve the $\sigma$-complete case of our question in full generality (the method applied in Th. 1 may not be suitable in general; the questions arising here may be related to the problem of the construction of a minimal $\sigma$-product for Boolean $\sigma$-algebras which seems to be still open (see [15], §38). We do have the following partial solution. (Let us recall that by a state on $L$ we mean a probability $\sigma$-additive measure on $L$. Obviously, a $\sigma$-OML may not possess any state – see, e.g., [14].)

**THEOREM 2.** *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of $\sigma$-OMLs and let $B$ be a Boolean $\sigma$-algebra. Let every $L_\alpha(\alpha \in I)$ possess a state. Then $\{L_\alpha | \alpha \in I\}$ has a $B$-envelope. Moreover, if every $L_\alpha(\alpha \in I)$ is concrete and if $B$ is concrete, then the $B$-envelope can be taken concrete, too.*

Proof. Let $D$ be a four-point Boolean algebra and let $K$ denote the horizontal sum of the collection $\{L_\alpha \mid \alpha \in I\} \cup \{D\}$. Since every $L_\alpha$ admits a state then so does $K$, too. Let us denote it by $s$, $s\colon K \to \langle 0,1\rangle$. As before, $K$ contains all the $L'_\alpha s$ ($\alpha \in I$). Let $(\Omega, \Sigma)$ be the Loomis–Sikorski representation of $B$ (see [15]) and let $q\colon \Sigma \to B$ be the corresponding factor mapping. Let now $\hat{L}$ be the set of all functions $f\colon \Omega \to K$ such that $s \cdot f\colon \Omega \to \langle 0,1\rangle$ be a $\sigma$-measurable mapping with respect to $\Sigma$ and Borel subsets of $\langle 0,1\rangle$. Making now use of the basic properties of measurable functions, one checks easily that $\hat{L}$ is a $\sigma$-OML. Moreover, $C(\hat{L}) = \Sigma$ for a two-valued function belongs to $\hat{L}$ if and only if it is a characteristic function of a set of $\Sigma$. If we now factorize $\hat{L}$ over the p-$\sigma$-ideal $I$ of "meagre sets" (i.e. $I = \{f \in \hat{L} \mid$ there is a set $J$, $J \in \Sigma$ such that $q(J) = 0 (\in B)$ and $f(a) = 0 (\in K)$ for any $a \in \Omega - J\}$), we obtain a $\sigma$-OML, some $L$. It is not difficult to check that $L$ possesses all the properties required in Th. 2 (the concreteness transfers to $L$ the same way as in Th. 1). The proof is complete. □

Let us finally take up the case of complete OMLs ($c$-OMLs). If the centre in question allows a (complete) Loomis–Sikorski parametrization, we have an analogy of the previous result. Unfortunately, not all complete Boolean algebras allow a (complete) Loomis–Sikorski parametrization (in fact, they do exactly if they are weakly distributive – see [15], p. 127 for the definition and other considerations; see also [11] for relevant investigations in OMLs).

**THEOREM 3.** *Let $\{L_\alpha \mid \alpha \in I)$ be a collection of complete OMLs and let $B$ be a complete weakly distributive Boolean algebra. Then there exists a complete $B$-envelope of $\{L_\alpha \mid \alpha \in I\}$. If moreover every $L_\alpha$ ($\alpha \in I$) is concrete and if $B$ is concrete, then the $B$-envelope can be taken concrete, too.*

The proof closely follows the pattern of the previous proof (it is sufficient to consider only step functions since every collection of partitions in $B$ admits a partition refinement).

In the realm of complete OMLs and quantum theories, an important position belongs to the OMLs of all projections in a von Neumann algebra. For these OMLs we have the following result. (The author would like to express his gratitude to L. Bunce for suggesting the use of the tensor product in the next theorem – see [3].)

**THEOREM 4.** *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of OMLs and let each of $L_\alpha$ ($\alpha \in I$) be an OML of all projections in a von Neumann algebra. Let $B$ be a complete (completely) set-representable Boolean algebra. Then there exists a $B$-envelope of $\{L_\alpha \mid \alpha \in I\}$. Moreover, this $B$-envelope can be required an OML of all projections in a von Neumann algebra.*

P r o o f. Suppose that $L_\alpha = \mathcal{P}(A_\alpha)$, where $\mathcal{A}_\alpha (\alpha \in I)$ is a von Neumann algebra. Let us denote by $\mathcal{A}$ the direct sum of $\mathcal{A}_\alpha$'s ($\alpha \in I$) in the category of von Neumann algebras. Then $\mathcal{P}(\mathcal{A})$ contains every $\mathcal{P}(\mathcal{A}_\alpha)$ (see [17], Chap. IV). Let $\mathcal{A}$ be viewed as an operator algebra for a Hilbert space $H$. Let $\mathcal{D}$ be the (von Neumann) algebra of all bounded operators on $H$. It is obvious that $\mathcal{P}(\mathcal{D})$ is a $\{0,1\}$-envelope of $\{L_\alpha \mid \alpha \in I\}$. Further, there is a commutative von Neumann algebra, $\mathcal{C}$, such that $B = \mathcal{P}(C)$. Let us now consider the "canonical" tensor product $\mathcal{C}\bar{\otimes}\mathcal{D}$ and let us denote by $L$ the projection OML of the latter von Neumann algebra. Thus, $L = \mathcal{P}(\mathcal{C}\bar{\otimes}\mathcal{D})$. If we denote by $\mathbb{C}$ the algebra of complex numbers, we have $C(L) = C\big(\mathcal{P}(\mathcal{C}\bar{\otimes}\mathcal{D})\big) = C\big(\mathcal{P}(\mathcal{C}\bar{\otimes}\mathbb{C})\big) = C\big(\mathcal{P}(C)\big) = B$ (see [17], Chap. IV for the proof of the latter three equalities). The proof is complete. □

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