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OPERATORS WHOSE EIGENVECTORS SPAN THE SPACE

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ABSTRACT. The operators on a complex separable Hilbert space H whose eigenvectors span H are investigated. They are shown to be hyperreflexive and some conditions under which such operators are reflexive are given. We recall an example of a non-reflexive operator whose commutant is reflexive.

1. Introduction

Let H be a complex separable Hilbert space, $\mathcal{B}(H)$ the algebra of all continuous linear operators on H and $T \in \mathcal{B}(H)$. We denote by $\{T\}'$ the *commutant* of T ($X \in \{T\}'$ if an only if $XT = TX$) and by $\{T\}'' = \bigcap\{\{X\}' \colon XT = TX\}$ the *bi-commutant* of T. A *contraction* means an operator $T \in \mathcal{B}(H)$ with norm $\|T\| \le 1$. By a *subspace* we always mean a closed linear subspace. A subspace $L \subset H$ is called *invariant* for $T \in \mathcal{B}(H)$ if $TL \subset L$. L is *hyperinvariant* for T if it is invariant for every $X \in \{T\}'$. If $\mathcal{A} \subset \mathcal{B}(H)$, then $\operatorname{Alg}\mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing $\mathcal{A}$ and the identity I. $\operatorname{Lat}\mathcal{A}$ denotes the set of all subspaces invariant for each $A \in \mathcal{A}$. $\operatorname{Lat}\mathcal{A}$ (with the operations $\cap$ and $\vee$ of the intersection and of forming the closed linear span, respectively) is a complete lattice. If $\mathcal{L}$ is a set of subspaces of H, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H) \colon \mathcal{L} \subset \operatorname{Lat}\{T\}\}$. Let us consider the following properties of an operator $T \in \mathcal{B}(H)$:

DEFINITION. Let $T \in \mathcal{B}(H)$. Then

(i) T is said to be *reflexive* if $\operatorname{Alg} T = \operatorname{Alg}\operatorname{Lat}\{T\}$,
(ii) T is said to be *hyperreflexive* if $\{T\}' = \operatorname{Alg}\operatorname{Lat}\{T\}'$.

In [1], a characterization of reflexive and hyperreflexive operators of the class C_0 was given and it was shown that every hyperreflexive C_0 contraction is reflexive and that the other implication does not hold. The problem whether hyper-

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reflexivity implies reflexivity remained open. Recently, an example of Larson and Wogen [5] was used to show [3] that the answer is negative. The purpose of the present paper is to give a little more detail of the solution of the above mentioned problem and to show its connections with Nevanlinna–Pick interpolation.

Results

We start with a simple sufficient condition for hyperreflexivity of an operator [3]:

LEMMA 1. *Let $T \in \mathcal{B}(H)$. If the closed linear span of all eigenvectors of T is H, then T is hyperreflexive.*

Proof. The idea of the proof goes back to Sarason [6]. If λ is an eigenvalue of T, then the eigenspace $\ker(\lambda - T)$ is hyperinvariant for T. It follows for every $A \in \operatorname{Alg}\operatorname{Lat}\{T\}'$ and for every eigenvector $h \in \ker(\lambda - T)$:

$$ATh = A(\lambda h) = \lambda Ah = TAh .$$

Since eigenvectors span H, $AT = TA$, i.e., T is hyperreflexive. □

A little more can be shown using this idea:

LEMMA 2. *Let $T \in \mathcal{B}(H)$ and let λ be an eigenvalue of T and let $A \in \operatorname{Alg}\operatorname{Lat}\{T\}$. Then there exists a complex number $a(\lambda)$ such that every eigenvector $h \in \ker(\lambda - T)$ is an eigenvector of A with the eigenvalue $a(\lambda)$.*

Consequently, if in addition eigenvectors of T span H, then $A \in \{T\}''$.

Proof. The one-dimensional space spanned by the eigenvector $h \in \ker(\lambda - T)$ is invariant for T and so also for A. Therefore, there exists a complex number a such that $Ah = a\,h$. If $g \in \ker(\lambda - T)$ is another eigenvector, then $\exists\, b, c \in C$ such that $Ag = b\,g$ and $A(h+g) = c\,(h+g) = a\,h + b\,g$. It is easy to prove that $b = c = a$. □

Since the space H is separable if eigenvectors of an operator $T \in \mathcal{B}(H)$ span H, then there exists a countable set $\{\lambda_n\}_{n=1}^{\infty}$ of eigenvalues of T such that the closed linear span of the corresponding eigenvectors $\bigvee \ker(\lambda_n - T) = H$. Let us assume (without loss of generality) that T is a contraction. To find some conditions under which T is reflexive, let us consider an operator $A \in \operatorname{Alg}\operatorname{Lat}\{T\}$ with norm $\|A\| \le 1$. We want to show that A can be approximated in the weak operator topology by polynomials in T, more precisely, we have to solve the following problem. Given an arbitrary $\varepsilon > 0$ and arbitrary n-tuples $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n$ of vectors in H, does there exist a polynomial p such that

$$\left|((A - p(T))x_i, y_i)\right| < \varepsilon, \qquad \text{for} \quad i = 1, 2, \dots, n \text{ ?} \tag{1}$$

According to Lemma 2, there exists a function mapping each eigenvalue λ_n of T onto an eigenvalue $a(\lambda_n)$ of A with the same eigenvectors. Therefore it is easy to construct a polynomial p satisfying (1) if all $x_1, x_2, \dots, x_n$ are eigenvectors of T. Indeed, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding eigenvalues of T it suffices to use Lagrange interpolation to obtain a polynomial mapping λ_i into $a(\lambda_i)$ and therefore satisfying $Ax_i = p(T)x_i$ for $i = 1, 2, \dots, n$.

Although every element $x \in H$ can be approximated by a finite linear combination of eigenvectors of T the above mentioned idea does not give the proof of reflexivity of T. The reason is that the norm of the operator $p(T)$ depends on the n-tuples $\lambda_1, \lambda_2, \dots, \lambda_n$ and $a(\lambda_1), a(\lambda_2), \dots, a(\lambda_n)$.

Since we consider now only contractions T we can use the H^∞ functional calculus (see [7]). Here H^∞ means the algebra of all complex functions bounded and analytic in the open unit disc $\{\lambda\colon |\lambda| < 1\}$. For every $u \in H^\infty$ $u(T) \in \operatorname{Alg} T$. Therefore, to prove that $A \in \operatorname{Alg}\operatorname{Lat}\{T\}$ it suffices to find a function $p \in H^\infty$ satisfying (1). The following theorem shows that this is related to the Nevanlinna–Pick interpolation problem.

THEOREM 3. *Let $T \in \mathcal{B}(H)$ and let $H = \bigvee_{n=1}^{\infty} \ker(\lambda_n - T)$. Let $A \in \operatorname{Alg}\operatorname{Lat}\{T\}$ and let $a(\lambda_n)$ $(n = 1, 2, \dots)$ be the eigenvalues of A as described in Lemma 2. If for all natural numbers N, the matrix*

$$\left(\frac{1 - a(\lambda_i)\overline{a(\lambda_j)}}{1 - \lambda_i\overline{\lambda_j}}\right)_{i,j=1,2,\dots,N} \tag{2}$$

is positive definite, then $A \in \operatorname{Alg} T$.

Proof. The well-known Nevanlinna–Pick theorem (see, e.g., [4]) asserts that the positive definiteness of the matrix (2) is equivalent to the existence of a function p_N analytic in the open unit disc and bounded by 1 (i.e., with $\|p_N\|_\infty = \sup\{|p_N(\lambda)|\colon |\lambda| < 1\} \le 1$) satisfying, for all $n = 1, 2, \dots, N$, $p_N(\lambda_n) = a(\lambda_n)$ and so also $p_N(T)x = Ax$ for $x \in \ker(\lambda_n - T)$. Since $\|p_N(T)\| \le \|p_N\|_\infty \le 1$, given an $h \in H$ and $\varepsilon > 0$, there exist a natural number N and a finite linear combination h_N of eigenvectors from eigensubspaces $\ker(\lambda_n - T)$ for $n = 1, 2, \dots, N$ such that $\|h - h_N\| < \varepsilon/2$. Then it holds

$$\|p_N(T)h - Ah\| = \|p_N(T)h - p_N(T)h_N + p_N(T)h_N - Ah_N + Ah_N - Ah\| \leq \\ \leq 2\|h - h_N\| < \varepsilon .$$

□

R e m a r k . According to a result of C a r l e s o n [2, Theorem 3], the condition

$$\inf_k \prod_{\substack{n=1 \\ k \neq n}}^{\infty} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_n} \lambda_k} \right| > 0 \tag{3}$$

implies that, for each bounded sequence of complex numbers $\{w_n\}$, there exists a function $f \in H^\infty$ such that $f(\lambda_n) = w_n$. This is sufficient for every $A \in \operatorname{Alg} \operatorname{Lat}\{T\}$ to be equal to $f(T)$ for a function $f \in H^\infty$. Put $w_n = a(\lambda_n)$ to be the eigenvalue of A given by Lemma 2. Observe that there exists a function $g \in H^\infty$ such that

$$f(\lambda) - w_n = (\lambda - \lambda_n)\, g(\lambda) \quad \text{and so} \quad f(T) = w_n I + g(T)(T - \lambda_n I).$$

$$\text{For } x \in \ker(T - \lambda_n) \qquad f(T)x = w_n x = Ax .$$

Since $H = \bigvee_{n=1}^{\infty} \ker(T - \lambda_n)$ this implies $f(T) = A$ and so T is reflexive.

But if the condition (3) holds, then T satisfies the assumption (4) of the following theorem which gives a simple sufficient condition for the reflexivity of a contraction.

THEOREM 4. *Let T satisfy the assumptions of Theorem 3. The following condition is sufficient for T to be reflexive:*

$$\sum_{n=1}^{\infty} (1 - |\lambda_n|) < \infty . \tag{4}$$

P r o o f . (4) implies that the Blaschke product $B(\lambda) = \prod_{n=1}^{\infty} \frac{\overline{\lambda_n}}{|\lambda_n|} \frac{\lambda_n - \lambda}{1 - \overline{\lambda_n}\lambda}$ converges, $B(T) = 0$ and so T is a contraction of class C_0. T is hyperreflexive and consequently also reflexive. □

Let us recall now what was the example of Larson and Wogen [5]. They constructed an operator T on a complex separable Hilbert space H with the following properties:

(i) The point spectrum of T consists of the sequence $\{\frac{1}{4^n}\}_{n=1}^{\infty}$.

(ii) The eigensubspaces $\ker(\frac{1}{4^n} - T)$ are one-dimensional and span H.

(iii) T is reflexive.

(iv) If K is any other complex separable Hilbert space with dimension at least one, then the operator $T \oplus 0$ on the space $H \oplus K$ is not reflexive.

They used this example to show that the orthogonal sum of two reflexive operators need not be reflexive. Since the eigenvectors of the operator $T \oplus 0$ span $H \oplus K$, this is also an example of an operator which is hyperreflexive but not reflexive.

Remark. It might be interesting to observe that both the operators T and $T \oplus 0$ do not satisfy any of the conditions (3) and (4).

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