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A QUANTUM LOGICS DESCRIPTION OF QUANTUM MEASUREMENTS

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ABSTRACT. Boolean powers of orthoalgebras are studied from the point of view of quantum measurements.

1. Introduction

In [3], quantum measurements are described in the frame of tensor product $H_1 \otimes H_2$ of Hilbert spaces H_1 and H_2, which are supposed to describe the measured quantum mechanical system and the measuring apparatus, respectively. An objection might arise that the measuring apparatus should be a macroscopic object, hence it should be described by methods of classical physics. In [21], there has been shown that such a description is possible in the frame of quantum logics (see [2, 20, 22]), if we use a Boolean power $L[B]$ to the description of the coupled system consisting of a quantum system described by a σ-orthomodular poset L and a measuring apparatus described by a complete Boolean algebra B. The Boolean power $L[B]$ has similar properties to those of a "tensor product" of orthomodular posets (see [1, 7, 17]). It is a well-known fact that the tensor product of orthomodular posets need not exist in general (see a counterexample in [11].) Therefore, a more general algebraic structure, namely an orthoalgebra, has been suggested to play the role of a quantum logic ([7, 8, 9, 10, 11, 12, 14]). In the frame of orthoalgebras, the existence of a tensor product is guaranteed, at least in the most important cases from the physical point of view.

The aim of the present paper is to extend Boolean powers of orthomodular posets to Boolean powers of orthoalgebras, and to investigate the latter concept from the point of view of quantum measurements.

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2. Basic properties of orthoalgebras

The following definition is due to A. Golfin [12] (see also [9, 14]).

DEFINITION 2.1. An *orthoalgebra* (OA) is a set L containing two special elements $0, 1$ and equipped with a partially defined binary operation $\oplus$ subject to the following conditions for all $p, q, r \in L$:

(i) (Commutativity.) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(ii) (Associativity.) If $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined, then $q \oplus r$ is defined, $p \oplus (q \oplus r)$ is defined, and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

(iii) (Orthocomplementation.) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(iv) (Consistency.) If $p \oplus p$ is defined, then $p = 0$.

If the hypotheses of (ii) are satisfied, we write $p \oplus q \oplus r$ for the element $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

We note that every orthomodular poset (OMP) is an orthoalgebra if we arrange that $p \oplus q$ is defined iff $p \perp q$, in which case $p \oplus q = p \vee q$. Especially, every Boolean algebra is an orthoalgebra.

Let L be an orthoalgebra and let $p, q \in L$. We say that p is *orthogonal* to q (written $p \perp q$) iff $p \oplus q$ is defined in L.

If there exists an element $r \in L$ such that $p \perp r$ and $q = p \oplus r$, we write $p \leq q$. The unique element q such that $p \perp q$ and $p \oplus q = 1$ is called the *orthocomplement* of p and it is denoted by p'. It then holds that $p \perp q$ iff $p \leq q'$. An orthoalgebra L with the partial order defined by $\leq$ and the orthocomplementation defined by $p \to p'$ is an orthocomplemented poset with 0 for a least and 1 for a greatest element, respectively. An analogue of the orthomodular identity is satisfied in L: if $p \leq q$, then $q = p \oplus (p \oplus q')'$ (see [9]).

The following theorem gives us the relations between orthoalgebras and orthomodular posets (see [9]).

THEOREM 2.2. *For an OA L, the following conditions are equivalent:*

(i) $(L, \leq, ', 0, 1)$ *is an OMP.*

(ii) *For $p, q, r \in L$, the conditions $p \perp q$, $p \perp r$ and $q \perp r$ imply that $(p \oplus q) \perp r$.*

(iii) *For $p, q \in L$, $p \perp q$ implies that $p \vee q$ exists.*

We note that if $p \perp q$ and $p \vee q$ exists in an OA L, then $p \vee q = p \oplus q$.

A subset L_1 of an OA L is called a *suborthoalgebra* of L if $0, 1 \in L$, L_1 is closed under the orthocomplementation map $p \to p'$, and, whenever $p, q \in L_1$ with $p \perp q$, it follows that $p \oplus q \in L_1$. Clearly, a suborthoalgebra L_1 of an OA L is an OA in its own right. As such, if L_1 is a Boolean algebra, we call it a Boolean subalgebra of L.

DEFINITION 2.3. Let L be an OA and let $C \subset L$.

(i) Elements of C are said to be *jointly compatible* (and C is called a compatible subset of L) if there is a Boolean subalgebra L_1 of L such that $C \subset L_1$.

(ii) Elements of C are said to be *jointly orthogonal* (and C is called an orthogonal subset of L) if the elements of C are jointly compatible and pairwise orthogonal.

If $p \leq q$, we define $q - p$ by $q - p = (q' \oplus p)'$. A finite set $D \subset L$ is called a *difference set* if either $D = \emptyset$ or there exists a strictly increasing sequence $p_0 \lneq p_1 \lneq p_2 \cdots \lneq p_n$ in L such that $D = \{p_k - p_{k-1} \colon k = 1, 2, \dots, n\}$. We define $\oplus D = p_n - p_0$ and $\oplus D = 0$ if $D = \emptyset$. Then $\oplus D$ is well defined, and D is a difference set iff D is a finite orthogonal set of nonzero elements (see [9]). Further, if D is a difference set and B is any Boolean subalgebra of L with $D \subset B$, then $\oplus D$ is effective as the least upper bound of D calculated in B. If C is a finite orthogonal set, then $D = C \setminus \{0\}$ is a difference set, and we define $\oplus C = \oplus D$. A difference set $E \subset L$ such that $\oplus E = 1$ is called a (finite) *partition* of unity in L.

It is easy to see that a finite set $\{a_1, a_2, \dots, a_n\}$ is a compatible subset of L iff there is a finite partition of unity $E \subset L$ such that for every a_i there is a subset $E_i \subset E$ with $a_i = \oplus E_i$ $(i = 1, 2, \dots, n)$. In particular, two elements p, q are compatible iff there are jointly orthogonal elements p_1, q_1, r such that $p = p_1 \oplus r$ and $q = q_1 \oplus r$.

3. σ-orthoalgebras, observables and states

Let L be an OA. If $(a_i)_{i=1}^{\infty}$ is a sequence of elements of L, we write $a_i \uparrow$ to denote that $a_i \leq a_{i+1}$ for $i = 1, 2, \dots$.

DEFINITION 3.1. We say that an OA L is a *σ-orthoalgebra* (σ-OA) if for any sequence $(a_i)_i$ in L such that $a_i \uparrow$ the supremum $a = \bigvee_{i=1}^{\infty} a_i$ exists in L. We write $a_i \uparrow a$.

Let $\mathcal{F}$ denote a σ-field of subsets of a nonempty set Ω. Especially, $(\Omega, \mathcal{F})$ may be equal to $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, where $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of all Borel subsets of the real line $\mathbb{R}$.

DEFINITION 3.2. A mapping $x : \mathcal{F} \to L$ is called an $(\Omega, \mathcal{F})$-*observable* on an OA L if

(i) $x(\emptyset) = 0$, $x(\Omega) = 1$.

(ii) $A, B \in \mathcal{F}$, $A \cap B = \emptyset \implies x(A) \perp x(B)$ and $x(A \cup B) = x(A) \oplus x(B)$.

(iii) $(A_i)_{i=1}^{\infty} \subset \mathcal{F}$ and $A_i \uparrow A$ imply $x(A_i) \uparrow x(A)$.

It is easy to see that for $A, B \in \mathcal{F}$, $A \subset B \implies x(A) \leq x(B)$. Indeed, $B = A \cup (A^c \cap B) \implies x(B) = x(A) \oplus x(A^c \cap B)$.

Proof of the following lemma is straightforward (see Theorem 2.2).

Lemma 3.3. *Let M be a subset of an OA L such that*

(i) $0 \in M$ *and* $a \in M \implies a' \in M$.

(ii) *If* p, q, r *are pairwise orthogonal elements contained in* M, *then* $p \oplus q \oplus r$ *exists in* L *and belongs to* M.

(iii) *If* $(a_i)_{i=1}^{\infty} \subset M$ *and* $a_i \uparrow a$, *then* $a \in M$.

Then M is a sub-σ-orthoalgebra of L which is a σ-orthomodular poset (σ-OMP).

Theorem 3.4. *Let L be a σ-OA. The range $\mathcal{R}(x) = \{x(E) \colon E \in \mathcal{F}\}$ of an $(\Omega, \mathcal{F})$-observable x on L is a Boolean sub-σ-algebra of L.*

Proof. We will prove that conditions of Lemma 3.3 are satisfied for $M = \mathcal{R}(x)$. Then, since $x \colon \mathcal{F} \to \mathcal{R}(x)$ is an observable onto the OMP $\mathcal{R}(x)$, $\mathcal{R}(x)$ is a Boolean algebra.

(i) Let $a \in \mathcal{R}(x)$, then $a = x(A)$ for some $A \in \mathcal{F}$. We have $1 = x(\Omega) = x(A \cup A^c) = x(A) \oplus x(A^c)$, hence $a' = x(A^c)$ belongs to $\mathcal{R}(x)$.

(ii) Let $a_i = x(A_i)$, $i = 1, 2, 3$ be pairwise orthogonal elements of $\mathcal{R}(x)$. Now $A_i = (A_1 \cap A_2 \cap A_3) \cup (A_i \cap A_j \cap A_k^c) \cup (A_i \cap A_j^c \cap A_k) \cup (A_i \cap A_j^c \cap A_k^c)$, $i, j, k \in \{1, 2, 3\}$. Owing to orthogonality of a_1, a_2, a_3, we obtain that $x(A_i) = x(A_i \cap A_j^c \cap A_k^c)$, $i, j, k \in \{1, 2, 3\}$. Hence $a_i = x(B_i)$, $i = 1, 2, 3$, where B_i, $i = 1, 2, 3$ are mutually disjoint. Since $(B_1 \cup B_2) \cap B_3 = \emptyset$, we obtain that

$$x(B_1 \cup B_2 \cup B_3) = x(B_1 \cup B_2) \oplus x(B_3) = x(B_1) \oplus x(B_2) \oplus x(B_3)$$

exists and belongs to $\mathcal{R}(x)$.

(iii) Let $a_i = x(A_i)$, $i = 1, 2, \dots, a_i \uparrow a$. Put $B_n = A_1 \cup A_2 \cup \dots \cup A_n$, $n = 1, 2, \dots$. We prove by induction that $a_n = x(B_n)$, $n = 1, 2, \dots$. Clearly, $a_1 = x(A_1) = x(B_1)$. Assume that $a_n = x(B_n)$. Then $B_{n+1} = B_n \cup A_{n+1} = A_{n+1} \cup (A_{n+1}^c \cap B_n) \implies x(B_{n+1}) = x(A_{n+1}) \oplus x(A_{n+1}^c \cap B_n)$. But $x(A_{n+1}^c \cap B_n) \le x(B_n) = a_n \le a_{n+1}$, and, on the other hand, $x(A_{n+1}^c \cap B_n) \le x(A_{n+1}^c) = a'_{n+1}$. Hence $x(B_{n+1}) = x(A_{n+1}) = a_{n+1}$. Now $B_n \uparrow \bigcup B_n = \bigcup A_i$. This implies that $a = \bigvee\limits_{i=1}^{\infty} a_i = x\Big(\bigcup\limits_{i=1}^{\infty} B_i\Big)$ belongs to $\mathcal{R}(x)$.

□

A *finitely additive measure* on an OA L is a mapping $m \colon L \to \mathbb{R}^+$ such that $m(a \oplus b) = m(a) + m(b)$ $(a, b \in L)$. A measure m is σ-additive if for any sequence $(a_i)_i$, $a_i \uparrow a$ we have $m(a_i) \uparrow m(a)$. A measure m is a state if $m(1) = 1$.

If x is an $(\Omega, \mathcal{F})$-observable on a σ-OA L and m is a σ-additive state on L, then $m \circ x$ is a probability measure on $\mathcal{F}$, the probability distribution of the observable x in the state m.

According to Theorem 3.4, notions concerning observables on OMPs can be transferred to observables on OAs, e.g., spectrum, functional calculus for compatible observables, etc.

4. Boolean powers of orthoalgebras

Definition 4.1. Let L be an OA and B a complete Boolean algebra. Define the set $L[B]$ as follows

$$L[B] = \{f\colon L \to B;\ f(\ell_1) \wedge f(\ell_2) = 0_B \quad \text{if} \quad \ell_1 \neq \ell_2 \quad \text{and} \quad \bigvee_{\ell \in L} f(\ell) = 1_B\}.$$

The set $L[B]$ is called *Boolean power* (or *Boolean extension*) of L (see [13]).

Theorem 4.2. *Boolean power $L[B]$ of an OA L is an orthoalgebra. Moreover, if L is an OMP, then $L[B]$ is an OMP.*

P r o o f . Let o and i be elements of L[B] defined by

$$o(x) = \begin{cases} 1_B\,, & \text{if } x = 0_L, \\ 0_B\,, & \text{if } x \neq 0_L, \end{cases} \qquad i(x) = \begin{cases} 1_B\,, & \text{if } x = 1_L, \\ 0_B\,, & \text{if } x \neq 1_L\,. \end{cases}$$

For $f, g \in L[B]$ define the relation $\perp$ by $f \perp g$ iff

$$\bigvee_{\substack{x,y \in L \\ x \perp y}} f(x) \wedge g(y) = 1_B\,,$$

and if $f \perp g$, define

$$f \oplus g(x) = \bigvee_{\substack{u,v \in L \\ u \oplus v = x}} f(u) \wedge g(v)\,.$$

We wish to prove that $L[B]$ with i and o as 1 and 0, respectively, and with the operation $\oplus$ is an OA. This can be done by straightforward checking conditions (i) – (iv) of Definition 2.1. We will sketch only the proof of associativity. Assume that $f, g, h \in L[B]$ and $f \perp g$, $f \oplus g \perp h$. Observe that

$$\begin{aligned} 1_B &= \bigvee_{\substack{x,y \in L \\ x \perp y}} f \oplus g(x) \wedge h(y) = \bigvee_{\substack{x,y,z \in L \\ x \oplus y = z}} f \oplus g(x) \wedge h(y) = \\ &= \bigvee_{\substack{x,y,z \in L \\ x \oplus y = z}} \Big(\bigvee_{\substack{u,v \in L \\ u \oplus v = x}} f(u) \wedge g(v) \Big) \wedge h(y) = \bigvee_{\substack{u,v,y,z \in L \\ (u \oplus v) \oplus y = z}} f(u) \wedge g(v) \wedge h(y) = \\ &= \bigvee_{\substack{u,v,y,z \in L \\ u \oplus (v \oplus y) = z}} f(u) \wedge g(v) \wedge h(y) = \bigvee_{\substack{u,x,z \in L \\ u \oplus x = z}} f(u) \wedge g \oplus h(x) \end{aligned}$$

which proves that $f \perp g \oplus h$. The relation $g \perp h$ follows from

$$1_B = \bigvee_{\substack{u,v,y,z \in L \\ (u \oplus v) \oplus y = z}} f(u) \wedge g(v) \wedge h(y) \leq \bigvee_{\substack{v,y \in L \\ v \perp y}} g(v) \wedge h(y)\,.$$

Similarly we prove that $(f \oplus g) \oplus h(x) = f \oplus (g \oplus h)(x)$ for every $x \in L$.

We note that $f \oplus g = i$ implies that $f(u) = g(u')$ $\forall u \in L$. Therefore, the orthocomplement f' of f is defined by $f'(u) = f(u')$, $u \in L$.

Now introduce the partial order in the usual way, i.e., $f \leq g$ if there is a $h \in L[B]$, $h \perp f$, $g = f \oplus h$. This gives $f \leq g$ iff $f \perp g'$, hence iff

$$1_B = \bigvee_{\substack{x,y \in L \\ x \perp y}} f(x) \wedge g'(y) = \bigvee_{\substack{x,y \in L \\ x \leq y}} f(x) \wedge g(y).$$

This definition agrees with the definition of partial order in Boolean powers of OMPs (see [5, 15, 21]), which entails the remaining part of the proof. □

As an OA L is a partial algebra, several kinds of morphisms can be considered. We will introduce the following definition of a monomorphism (see [8]), which is suitable for our purposes.

DEFINITION 4.3. Let L_1, L_2 be orthoalgebras. The mapping $h\colon L_1 \to L_2$ is called a *monomorphism* if

(i) $h(1) = 1$,
(ii) $a \perp b$ iff $h(a) \perp h(b)$ and $h(a \oplus b) = h(a) \oplus h(b)$.

LEMMA 4.4. *Let $h\colon L_1 \to L_2$ be a monomorphism. Then*

(i) $h(a') = h(a)'$,
(ii) $h(a) = h(b) \implies a = b$.

Proof. (i) By definition, $a \perp a' \implies h(a) \perp h(a')$ and $1 = h(a \oplus a') = h(a) \oplus h(a')$, which implies $h(a') = h(a)'$. (ii) $h(a) = h(b) \implies h(a) \perp h(b)'$ and $h(b') = h(b)'$ give $a \perp b'$, i.e., $a \leq b$. By symmetry, $a = b$. □

Let $L[B]$ be a Boolean power of an OA L. Define the mappings

$$\lambda\colon L \to L[B],$$
$$\lambda(\ell)\colon L \to B,$$
$$\lambda(\ell)(x) = \begin{cases} 1_B, & \text{if } \ell = x, \\ 0_B, & \text{if } \ell \neq x, \end{cases}$$

and

$$\beta\colon B \to L[B],$$
$$\beta(b)\colon L \to B,$$
$$\beta(b)(x) = \begin{cases} b, & \text{if } x = 1_L, \\ b^c, & \text{if } x = 0_L, \\ 0, & \text{otherwise}. \end{cases}$$

THEOREM 4.5. *Let $L[B]$ be a Boolean power of an OA L. The mappings $\lambda\colon L \to L[B]$ and $\beta\colon B \to L[B]$ are monomorphisms which preserve all existing suprema (infima) in L and B, respectively.*

Proof. It suffices to prove that $\lambda\colon L \to L[B]$ is a monomorphism of orthoalgebras. The rest of the proof is then analogical to that for OMPs (see [21]).

Clearly, $\lambda(0_L) = o$, $\lambda(1_L) = i$. Let $p, q \in L$, then

$$\lambda(p)(x) \wedge \lambda(q)(y) = \begin{cases} 1_B, & \text{if } x = p,\ y = q, \\ 0_B, & \text{otherwise}, \end{cases}$$

hence $\lambda(p) \perp \lambda(q)$ iff $p \perp q$. This proves that λ is a monomorphism. □

Similarly as in the Boolean powers of OMPs, every element $f \in L[B]$ can be written in the form

$$f = \sum_i \lambda(\ell_j) t_j ,$$

where $(t_j)_j$ is a partition of unity in B, and $(\ell_j)_j$ are elements of L, in the sense that, for any $x \in L$,

$$f(x) = \bigvee_j \lambda(\ell_j)(x) \wedge t_j .$$

If the elements $(\ell_j)_j$ are mutually different, the above representation is unique. If $f, g \in L[B]$ have the representations $f = \sum\limits_j \lambda(\ell_j) t_j$, $g = \sum\limits_i \lambda(k_i) s_i$, then it follows that $f \le g$ iff $t_j \wedge s_i \ne 0 \implies \ell_j \le k_i$. Therefore, if $(w_s)_s$ is the common refinement of the partitions $(t_j)_j$ and $(s_i)_i$, so that $f = \sum\limits_s \lambda(\ell_s) w_s$, $g = \sum\limits_s \lambda(k_s) w_s$, we obtain $f \oplus g = \sum\limits_s \lambda(\ell_s \oplus k_s) w_s$, $f \vee g = \sum\limits_s \lambda(\ell_s \vee k_s) w_s$, $f \wedge g = \sum\limits_s \lambda(\ell_s \wedge k_s) w_s$, provided that the corresponding elements $\ell_s \oplus k_s$, $\ell_s \vee k_s$, $\ell_s \wedge k_s$ exist in L. In particular, $\lambda(\ell) \wedge \beta(b) = \lambda(\ell) b + \lambda(0_L) b^c$ exists in $L[B]$ for any $\ell \in L$ and $\beta \in B$. Moreover, it is easy to see that

$$\lambda(\ell) \wedge \beta(b) \oplus \lambda(\ell) \wedge \beta(b^c) = \lambda(\ell)$$

and

$$\lambda(\ell) \wedge \beta(b) \oplus \lambda(\ell') \wedge \beta(b) = \beta(b) .$$

This shows that the elements $\lambda(\ell) \wedge \beta(b)$, $\lambda(\ell) \wedge \beta(b^c)$ and $\lambda(\ell') \wedge \beta(b)$ are jointly orthogonal, and $\lambda(\ell)$ and $\beta(b)$ are compatible elements in $L[B]$ for every $\ell \in L$ and $b \in B$.

Let m be a state on L and μ a completely additive state on B. Define

$$m \otimes \mu\Big(\sum_j \lambda(\ell_j) t_j\Big) = \sum_j m(\ell_j) \mu(t_j) .$$

It is easy to prove that $m \otimes \mu$ is a finitely additive state on $L[B]$. Moreover,

$$m \otimes \mu\big(\lambda(\ell) \wedge \beta(b)\big) = m(\ell)\mu(b)\,.$$

The state $m \otimes \mu$ is called a product state.

Summarizing, we obtain the following result.

THEOREM 4.6. *Let L be an OA and let B be a complete Boolean algebra. The Boolean power $L[B]$ has the following properties:*

(i) *$L[B]$ is an orthoalgebra.*

(ii) *The mappings $\lambda\colon L \to L[B]$, $\beta\colon B \to L[B]$ are monomorphisms preserving all existing suprema and infima.*

(iii) *$\lambda(\ell)$ is compatible with $\beta(b)$ for any $\ell \in L$ and $b \in B$. Moreover, $\lambda(\ell) \wedge \beta(b)$ exists in $L[B]$ and is equal to $o \in L[B]$ iff $\ell = 0_L$ or $b = 0_B$.*

(iv) *If m is a state on L and μ is a completely additive state on B, then there is a finitely additive state $m \otimes \mu$ on $L[B]$ such that $m \otimes \mu\big(\lambda(\ell) \wedge \beta(b)\big) = m(\ell) \wedge \mu(b)$.*

In what follows, the following proposition will be useful.

PROPOSITION 4.7. *Let L be a σ-OA, B a complete Boolean algebra. For any element $b \in B$ and any state s on $L[B]$, the mapping $s_{\beta(b)}\colon L[B] \to [0,1]$ such that*

$$s_{\beta(b)}(f) = s(\beta(b) \wedge f), \quad f \in L[B]\,,$$

is a finitely additive measure on $L[B]$. Moreover, if the restriction $s \circ \lambda$ of s to $\lambda(L)$ is σ-additive, the same holds for the restriction $s_{\beta(b)} \circ \lambda$ of $s_{\beta(b)}$.

Proof. The form of $\beta(b)$ guarantees the existence of $\beta(b) \wedge f$ for every $b \in B$ and $f \in L[B]$, namely

$$\beta(b) \wedge f = \sum_i \lambda(\ell_i)t_i \wedge b + \lambda(0)b^c\,,$$

where $f = \sum_i \lambda(\ell_i)t_i$. From this it easily follows that $\beta(b) \wedge (f \oplus g) = \big(\beta(b)\wedge\big)f \oplus \big(\beta(b) \wedge g\big)$ provided that $f \oplus g$ exists. Moreover, if $(\ell_i)_i \subset L$ and $\ell_i \uparrow \ell$, then from $\beta(b) \wedge \lambda(\ell_i) = \lambda(\ell_i)b + \lambda(0)b^c$, $i = 1,2,\dots$ and from $\lambda(\ell) = \bigvee_i \lambda(\ell_i)$ we get $\beta(b) \wedge \lambda(\ell) = \bigvee_i \beta(b) \wedge \lambda(\ell_i)$. This yields the desired result. □

In [10], Pták's sums of orthoalgebras are studied (that is, bounded Boolean powers, see [5, 15, 18]). Recall that a bounded Boolean power $L \oplus B$ of L with respect to a Boolean algebra B (not necessarily complete) consists of all functions $f\colon L \to B$ with finite range and such that $f(\ell_1) \wedge f(\ell_2) = 0_B$ if $\ell_1 \neq \ell_2$ and $\bigvee_{\ell \in L} f(\ell) = 1_B$. We note that the methods of our proof apply also to bounded Boolean powers to obtain that $L \oplus B$ is an orthoalgebra. A similar result was obtained using another technique in [10], where it was also proved that $L \oplus B$ is a special kind of tensor product of orthoalgebras introduced in [8]. We note that the measurement theory introduced in the next paragraph can be also formulated in the frame of bounded Boolean powers. An advantage of using full Boolean powers is that the Boolean power $L[B]$ with respect to an atomic Boolean algebra B is isomorphic to the direct product of copies of L indexed by the set of atoms of B, which is a σ-OA provided that L is a σ-OA. In this case (if the cardinality of the set of atoms of B is not too great, see [20]) the set of all states on $L[B]$ consists exactly of all convex combinations of the product states.

5. Quantum measurements

In the quantum logic approach, every physical system is described by a quantum logic, which mathematically represents the set of all experimentally verifiable propositions of the system or, equivalently, the set of all random events of the system.

Let $\mathcal{S}$ be a microscopic system described by a quantum logic L and let X be an observable on L. Measurement of X requires specifying a measuring apparatus $\mathcal{A}$ and an observable $X_{\mathcal{A}}$ of $\mathcal{A}$. The measuring apparatus should be a macroscopic object, and we will assume that it is described by a complete Boolean algebra B. In general, X is an $(\Omega, \mathcal{F})$-observable on L, while $X_{\mathcal{A}}$ is a $(\Omega_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$-observable on B. Therefore we need a measurable function $\xi\colon \Omega_{\mathcal{A}} \to \mathcal{A}$ to relate Ω to $\Omega_{\mathcal{A}}$. We assume that the coupled system $\mathcal{S}+\mathcal{A}$ is described by the Boolean power $L[B]$. We also assume that the possible initial (σ-additive) states of $\mathcal{S}$ form a convex set M and the possible initial (completely additive) states of the measuring apparatus $\mathcal{A}$ form a convex set $M_{\mathcal{A}}$. For physical states on $L[B]$ we will consider the set $\mathcal{P}$ consisting of all convex combinations of product states $m \otimes m_{\mathcal{A}}$ with $m \in M$ and $m_{\mathcal{A}} \in M_{\mathcal{A}}$. Now if m is an initial state of $\mathcal{S}$ and $m_{\mathcal{A}}$ is an initial state on $\mathcal{A}$, then, since we assume that $\mathcal{S}$ and $\mathcal{A}$ are independent before measurement, the initial state of the coupled system should be $m \otimes m_{\mathcal{A}}$. In agreement with [3, 21], a measurement is a 5-tuple $\mathcal{M} = (B, X_{\mathcal{A}}, m_{\mathcal{A}}, \xi, V)$, where V is a transformation of the state space $\mathcal{P}$ which preserves convex combinations. The property which classifies the 5-tuple $\mathcal{M}$ as a measurement is the following equality:

$$m\big(x(E)\big) = V(m \otimes m_{\mathcal{A}}) \circ \beta\big(X_{\mathcal{A}}(\xi^{-1}(E))\big)$$

for all $E \in \mathcal{F}$ and all initial states m on L. If also

$$m\big(X(E)\big) = V(m \otimes m_{\mathcal{A}}) \circ \lambda\big(x(E)\big)$$

for all $E \in \mathcal{F}$ and all initial states m on L, then $\mathcal{M}$ is called a measurement of first kind.

Clearly, the transformation V can be extended to a transformation of the positive cone $K_{\mathcal{P}} = \{\alpha m\colon \alpha \geq 0,\ m \in \mathcal{P}\}$ by positive homogeneity.

We can also introduce the notion of an instrument $I_{\mathcal{M}}$ to the measurement $\mathcal{M}$ through

$$I_{\mathcal{M}}(E)(m) = V(m \otimes m_{\mathcal{A}})_{\beta(X_{\mathcal{A}} \circ \xi^{-1}(E))} \circ \lambda ,$$

where $E \in \mathcal{F}$ and m is an initial state on L. Then $I_{\mathcal{M}}(E)(m)$ is interpreted as the nonnormalized final state of $\mathcal{S}$ on the condition that the measurement leads to a result in E. A measurement $\mathcal{M}$ is called repeatable if the instrument $I_{\mathcal{M}}$ satisfies

$$I_{\mathcal{M}}(E)\big(I_{\mathcal{M}}(F)(m)\big)(1_L) = I_{\mathcal{M}}(E \cap F)(m)(1_L)$$

for all $E, F \in \mathcal{F}$ and all states $m \in M$.

As we can see from the preceding paragraphs, the above definitions make sense also in the case when the quantum logic L is a σ-orthoalgebra. Since the measuring apparatus is a classical object described by a Boolean algebra, no problems with the objectification of the measurements occur (compare [3]).

Recall that a Boolean algebra B satisfies the *countable chain condition* (ccc) if every chain in B is at most countable. The following result states that a measurement for an observable whose range satisfies ccc always exists.

THEOREM 5.1. *Let X be an observable on an OA L such that the range $\mathcal{R}(X)$ of X satisfies ccc. Then there exists a measurement of X.*

Proof. Let $(\Omega, \mathcal{F})$ be the value space of X, i.e., $X\colon \mathcal{F} \to L$. Let $\mathcal{N}$ denote the σ-ideal of X–null sets in $\mathcal{F}$, i.e., $\mathcal{N} = \{E \in \mathcal{F}\colon X(E) = 0\}$, and let $B = \mathcal{F}/\mathcal{N}$ be the quotient of $\mathcal{F}$ with respect to $\mathcal{N}$. Then B is isomorphic with $\mathcal{R}(X)$, hence B satisfies ccc, so that it is a complete Boolean algebra. Now define $X_{\mathcal{A}}\colon \mathcal{F} \to B$ by $X_{\mathcal{A}}(F) = [F]$, where $[F]$ is the equivalence class in $\mathcal{F}/\mathcal{N}$ containing F. Let $(E_i)_i$ be any countable partition of Ω consisting of elements of $\mathcal{F}$. For a (σ-additive) state m on L define the measures μ_i on B by $\mu_i([E]) = m\big(X(E_i \cap E)\big)$, $i = 1, 2, \dots$. Since B satisfies ccc, it can be easily seen that μ_i is a well defined completely additive measure on B for every i. Let $(m_i)_i$ be any sequence of states on L. Put further $\xi = id$, and let $m_{\mathcal{A}}$ be any state on B. Finally, define

$$V(m \otimes m_{\mathcal{A}}) = \sum_i m\big(X(E_i)\big)\ m_i \otimes \frac{\mu_i}{m\big(X(E_i)\big)} .$$

We assume that M and $M_{\mathcal{A}}$ are the sets of all states on L and B, respectively. In is easy to check that V can be extended by convexity to a convex transformation of the set $\mathcal{P}$. For every $E \in \mathcal{F}$ we have

$$V(m \otimes m_{\mathcal{A}})(\beta(X_{\mathcal{A}}(E)) = \sum_i m\big(X(E_i \cap E)\big) = m\big(X(E)\big) ,$$

hence $(B, X_{\mathcal{A}}, m_{\mathcal{A}}, id, V)$ is a measurement for X. □

We note that if we describe the compound system $\mathcal{S} + \mathcal{A}$ by a bounded Boolean power $L \oplus B$, the condition that the range $\mathcal{R}(X)$ of X satisfies ccc can be omitted, and we obtain that a measurement exists for every observable.

The above measurement need be neither of first kind nor repeatable. The existence of measurements which are repeatable or of first kind is an open problem, in general. Our next statement which is a generalization of the results in [21], gives a partial answer. We will need the following definition.

DEFINITION 5.2. Let L be a σ-OMP and m a state on L. Let $b \in L$ be such that $m(b) \neq 0$. A function $p_m(./b)$ on L satisfying

(i) $p_m(./b)$ is a state on L and $p_m(b/b) = 1$,

(ii) $p_m(a/b) = \frac{m(a \wedge b)}{m(b)}$ provided that a is compatible with b

is called a *conditional state* on L (with respect to m, b).

We say that L admits conditional states with respect to a set M of states if a unique $p_m(\cdot/b)$ exists for every state $m \in M$ and every $b \in L$ such that $m(b) \neq 0$.

Projection lattices of von Neumann algebras with no I_2 factor as direct summand ([4]) and some projection lattices of GL-spaces ([6]) are known examples of OMLs admitting conditional states.

THEOREM 5.3. *Let L be a σ-OMP which admits conditional states with respect to a convex set M of states and let X be an observable on L with discrete spectrum. Then there exists a repeatable first-kind measurement of X.*

P r o o f . We restrict the set of possible initial states of the measured system to M.

Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be the spectrum of X. Let $\mathcal{F}$ be the complete Boolean algebra of all subsets of Ω. Clearly, X can be considered as an $(\Omega, \mathcal{F})$-observable on L. Further, put $B \equiv \mathcal{F}$, and consider the observable $X_{\mathcal{A}}\colon \mathcal{F} \to B$ defined by $X_{\mathcal{A}}(E) = E$, $E \in \mathcal{F}$. Let $\xi \equiv id$ and let $m_{\mathcal{A}}$ be any completely additive state on B. Finally, for a state $m \in M$, define

$$V(m \otimes m_{\mathcal{A}}) = \sum_{\{i:\ m(b_i) \neq 0\}} m(b_i) p_m(\cdot/b_i) \otimes \mu_i$$

where $b_i = X(\{\omega_i\})$ and μ_i is a state on B concentrated at ω_i. Extend V, by convexity, to the whole $\mathcal{P}$. It is easy to check that $\mathcal{M} = (B, X_{\mathcal{A}}, m_{\mathcal{A}}, \xi, V)$ is a repeatable measurement of X. □

We note that the measurement in the above theorem corresponds to the Lüders measurement defined in [3].

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