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OBSERVABLES AND EXPECTATION ON PTÁK'S SUM

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ABSTRACT. A sum of a Boolean algebra and a quantum logic has been defined by P. Pták [2] and is studied by V. Janiš, Z. Riečanová [4], V. Janiš [3], V. Janiš, O. Nánásiová [5], O. Nánásiová [6], [11], C. A. Drossos [7] etc. It was shown that there is a special case when this structure is a direct product [4], [5]. C. A. Drossos [7] has studied the connection between this structure and a bounded Boolean power. The sum is a free product iff the quantum logic is a Boolean algebra, too [11]. A representation of a state on Pták's sum with the function of conditional probability on quantum logic is given. Further, we describe a conditional expectation of an observable on Pták's sum using the conditional expectation of an observable on the quantum logic with range in the center of the quantum logic. An example of this representation is given.

1. Introduction

Let L be a *quantum logic* (briefly q.l.). In this paper, we consider the quantum logic as an *orthomodular lattice.* Precisely, L is a partially ordered set with the first and the last elements 0 and 1 respectively, with the *orthocomplementation* $\perp : L \to L$ such that

(1) $(a^{\perp})^{\perp} = a$, for $a \in L$;
(2) $a \leq b$ implies $a^{\perp} \geq b^{\perp}$, $a, b \in L$;
(3) for all $a \in L$ we have $a \vee a^{\perp} = 1$;
(4) for any $a_1, ..., a_n \in L$ there exists $\bigvee_{i=1}^{n} a_i \in L$;
(5) if $a \leq b$, then $b = a \vee (b \wedge a^{\perp})$ $(a, b \in L)$.

Two elements $a, b \in L$ are *orthogonal* if $a \leq b^{\perp}$, and $a, b \in L$ are *compatible* $(a \leftrightarrow b)$ if $a = (a \vee b) \wedge (a \vee b^{\perp})$. If $a_i \in L$ for any $i = 1, 2, \ldots, n$ and $b \in L$ is such that $a_i \leftrightarrow b$ for all i, then $b \leftrightarrow \bigvee_i a_i$ and $b \wedge \left(\bigvee_i a_i\right) = \bigvee_i (a_i \wedge b)$ [1], [7].

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Let M be a subset of L and $a \in L$, $a \neq 0$. We shall say that M is *partially compatible with respect to* a (abb. as M is p.c. $[a]$) if the following is true:

(1) for all $b \in M$ we have $b \leftrightarrow a$ ($M \leftrightarrow a$; in symbols);
(2) for all $b, c \in M$ we have $b \wedge a \leftrightarrow c \wedge a$ (i.e., $M \wedge a = \{b \wedge a : b \in M\}$ is a compatible set).

A subset $L_0 \subseteq L$ is a *sublogic* of L if, for any $a \in L$, we have $a^\perp \in L$ and, for any $a_1, \dots, a_n \in L$, $\bigvee_i a_i \in L$. If for any $a, b \in L$, $a \leftrightarrow b$, then L is a *Boolean algebra*. In the following we shall pick up $C(L)$ the *center of* L, i.e., $C(L) = \{a \in L;\ a \leftrightarrow b \text{ for any } b \in L\}$ ([1], [8]).

A *state* on L is a map m from L into the interval $[0,1]$ on the real line such that

(1) $m(1) = 1$;
(2) $m(\bigvee_i a_i) = \sum_i m(a_i)$ if $a_i \leq a_j^\perp$, for all $i \neq j$ $(i, j = 1, 2, \dots, n)$.

If L is a quantum logic, then $S(L)$ will denote the set of all states on L. For $S \subseteq S(L)$ we shall say that (L, S) is a *quite full system* (q.f.s.) if $\{m \in S;\ m(a) = 1\} \subseteq \{m \in S;\ m(b) = 1\}$ implies $a \leq b$ [8].

Let L_1, L_2 be logics. Then a mapping $f\colon L_1 \to L_2$ is called a *homomorphism*, if

(1) $f(1) = 1$;
(2) $f(a^\perp) = f(a)^\perp$;
(3) $f(a \vee b) = f(a) \vee f(b)$ for $a, b \in L_1$ such that $a \leq b^\perp$.

If $L_1 = B(\mathbb{R}^1)$, then a homomorphism f is called an observable on L_2. Let x be an observable on L. The *spectrum* of x is defined as the smallest closed set $\sigma(x)$ such that $x(\sigma(x)) = 1$.

The set $R(f) = \{f(a);\ a \in L_1\}$ is called the *range of the homomorphism f.* Two homomorphisms $h\colon L_1 \to L$, $g\colon L_2 \to L$ are called *compatible* if, for any $a \in L_1$ and for any $b \in L_2$, $h(a) \leftrightarrow g(b)$ (where L_1, L_2, L are quantum logics).

If a mapping $f\colon L_1 \to L$ is an injective homomorphism and $f^{-1}\colon f(L_1) \to L_1$ is a homomorphism, then f is called an *embedding* ([8]).

Let L, Q be quantum logics. Let m be a state on L and h be a homomorphism from Q to L. It is clear that a map m from Q into L such that $m_h(a) = m(h(a))$ is a state on Q.

DEFINITION 1.2. ([2]) Let B and L be a Boolean algebra and a quantum logic, respectively. Then $B \oplus L$ is a quantum logic with the following properties:

(1) there exist embeddings $f_0\colon B \to B \oplus L$, $f_1\colon L \to B \oplus L$ such that $f_0(a) \wedge f_1(b) = 0$ iff $a = 0$ or $b = 0$;
(2) there is no proper sublogic L containing $f_0(B) \cup f_1(L)$;

(3) for each couple of states $m_0 \in S(B)$, $m_1 \in S(L)$ there exists a state $\mu \in S(B \oplus L)$ such that $\mu(f_0(a)) = m_0(a)$ for each $a \in B$ and $\mu(f_1(b)) = m_1(b)$ for any $b \in L$ $(\mu = (m_0, m_1))$.

This structure is known as *Pták's sum*. In the following, we will mention only the main properties of this structure. For any $a \in B \oplus L$ there exists an orthogonal partition of 1, $c_1, \dots, c_n$, from B and elements $a_1, \dots, a_n \in L$ such that $\boldsymbol{a} = \bigvee_i (f_0(c_i) \wedge f_1(a_i))$. We can write $\boldsymbol{a}$ as the "vector" $\boldsymbol{a} = [(c_1, a_1), \dots, (c_n, a_n)]$ and $f_0(c) = [(c,1), (c^{\perp}, 0)]$, $f_1(a) = [(1,a)]$.

Proposition 1.1. ([11]) *Let L, A be quantum logics, B a Boolean algebra and $B \oplus L$ the Ptak's sum. Then a map $z\colon B \oplus L \to A$ is a homomorphism iff there exist two homomorphisms h, g such that $h\colon B \to A$, $g\colon L \to A$ and $h(a) \leftrightarrow g(b)$, for any $a \in B$ and any $b \in L$ where $h = f_0 \circ z$, $g = f_1 \circ z$.*

Let L be a quantum logic. Let us denote by $S(L)$ the set of all states on L. Let B be a Boolean algebra. If $M_0 \subseteq S(B)$ and $M_1 \subseteq S(L)$, then $M_0 \times M_1 \subseteq S(B \oplus L)$ in the sense that $\mu \in M_0 \times M_1$ iff there exist $m_0 \in M_0$, $m_1 \in M_1$ with $\mu = (m_0, m_1)$. It is clear that $S(B) \times S(L) \subseteq S(B \oplus L)$ [11].

Proposition 1.2. ([11]) *Let $B \oplus L$ be a Pták's sum and h be a map from B to L. A map $g\colon B \oplus L \to L$ which is defined as $g([(c_1, a_1), \dots, (c_n, a_n)]) = \bigvee_i h(c_i) \wedge a_i$ is a homomorphism iff h is a homomorphism from B to $C(L)$.*

Proposition 1.3. ([11]) *Let L be quantum logic such that $C(L) \neq \{0,1\}$ and B be a Boolean algebra. Let $m \in S(L)$ and h be a homomorphism from B to $C(L)$ such that there exist $c \in B$ with $m(h(c)) \neq 1, 0$. Then there exist two states $\alpha, m \in S(B \oplus L)$ such that $\alpha \neq m$ but $\alpha/f_1(L) = m/f_1(L)$ and $\alpha/f_0(B) = m/f_0(B)$.*

Proposition 1.4. ([11]) *Let B be a Boolean algebra and let (L, M) be q.f.s., where M is a convex set of states.*

(1) *If there exists a homomorphism h from B to L such that $R(h) \neq \{0,1\}$, then there exist states $\mu, \alpha \in S(B \oplus L)$ such that $\mu/f_0(B) = \alpha/f_0(B)$, $\mu/f_1(L) = \alpha/f_1(L)$ but $\mu \neq \alpha$.*
(2) *$C(L) = \{0,1\}$ iff for any homomorphism h from B to $C(L)$ and and for any state $m \in M$, we have*

$$\sum_{i=1}^{n} m(h(c_i)) m(a_i) = \sum_{i=1}^{n} m(h(c_i) \wedge a_i),$$

where $[(c_1, a_1), \dots, (c_n, a_n)] \in B \oplus L$.

2. States as conditional probability

Let L be a quantum logic. Let $\mathbb{L} = L \times L \times \cdots$ be the direct product of copies of L. With coordinatewise defined partial order and orthocomplementation $\mathbb{L}$ becomes a quantum logic. Let us denote $L(a) = \{(a_1, a_2, a_3, \ldots);\ a_i \in L$ for $i = 1, 2, \ldots$ and $a_i \neq a$ only for finitely many $i.\}$ and $\mathfrak{L} = \bigcup\limits_{a \in L} L(a)$.

Example 2.1. Let B be a Boolean algebra with a countable set of atoms $\{b_1, b_2, \ldots\}$ such that for any $b \in B$ there exist a number n such that $b = \bigvee\limits_{i=1}^{n} b_{j_i}$ or $b^{\perp} = \bigvee\limits_{i=1}^{n} b_{j_i}$. Let L be quantum logic. Let $B \oplus L$ be Pták's sum. Then there is an isomorphism between $B \oplus L$ and $\mathbb{L}$.

Indeed, if $(a_1, a_2, a_3, \ldots) \in L_1(a)$, then there exits n such that for any $n \leq i$ $a_i = a$. In the following, we will assume $\{b_1, b_2, \ldots\}$ as ordered set of all atoms. Let h be a map such that

$$h(a_1, a_2, a_3, \ldots) = \left[(b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n), \left(\left(\bigvee_{i=1}^{n} b_i\right)^{\perp}, a\right)\right].$$

(For any ordered set of all atoms $\{b_{i1}, b_{i2} \ldots,\}$ we get another map h.)

We will prove that h is an isomorphism. It is clear that $h(1, 1, \ldots) = [(1, 1)]$ and $h(0, 0, \ldots) = [(1, 0)]$. Let $\boldsymbol{a} \in L(a)$, $\boldsymbol{c} \in L(c)$. It means that

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n, a, a, \ldots)$$

and

$$\boldsymbol{c} = (c_1, c_2, \ldots, c_k, c, c, \ldots).$$

Let $k \leqq n$. Then

$$\boldsymbol{a} \vee \boldsymbol{c} = (a_1 \vee c_1,\ a_2 \vee c_2, \ldots,\ a_k \vee c_k,\ a_{k+1} \vee c, \ldots,\ a_n \vee c,\ a \vee c, \ldots).$$

Now we have

$$\begin{aligned} h(\boldsymbol{a} \vee \boldsymbol{c}) &= \left[(b_1, a_1 \vee c_1), \ldots, (b_k, a_k \vee c_k), \ldots, (b_n, a_n \vee c), \left(\left(\bigvee_{i=1}^{n} b_i\right)^{\perp}, a \vee c\right)\right] \\ &= \left[(b_1, a_1), \ldots, (b_k, a_k), \ldots (b_n, a_n), \left(\left(\bigvee_{i=1}^{n} b_i\right)^{\perp}, a\right)\right] \vee \\ &\quad \vee \left[(b_1, c_1), \ldots, (b_k, c_k), (b_{k+1}, c), \ldots, (b_n, c), \left(\left(\bigvee_{i=1}^{n} b_i\right)^{\perp}, c\right)\right] \\ &= h(\boldsymbol{a}) \vee h(\boldsymbol{c}). \end{aligned}$$

It is known that $\boldsymbol{a}^{\perp} = (a_1^{\perp}, a_2^{\perp}, \dots, a_n^{\perp}, a^{\perp}, \dots)$, and therefore

$$h(\boldsymbol{a})^{\perp} = h(\boldsymbol{a}^{\perp}).$$

If $\boldsymbol{a} \neq \boldsymbol{c}$, then $h(\boldsymbol{a}) \neq h(\boldsymbol{c})$. From this we get that h is an embedding.

Let $\big[(d_1, a_1), \dots, (d_n, a_n)\big] \in B \oplus L$. There is exactly one i such that $d_i = \bigvee\limits_{j=1}^{\infty} b_{ij}$ where $b_{ij} \in \{b_1, b_2, \dots\}$. Then for any $k \neq i$ $d_k = \bigvee\limits_{r=1}^{n_k} b_{kr}$. Now we have a map $g\big([(d_1, a_1), \dots, (d_n, a_n)]\big) = (c_1, c_2, \dots)$, where $c_j = a_k$ if $b_j \leqslant d_k$. It is easy to show that g is embedding and $g = h^{-1}$.

Definition 2.1. [12] Let L be a q.l. and $L_c \subset L \setminus \{0\}$. Let $p\colon L \times L_c \to \mathbb{R}^1$ satisfies:

(1) for any $b \in L_c$, $p(./b)$ is a state on L and $p(b/b) = 1$;

(2) if $p(c/1) = 1$ and b, $(c \vee b^{\perp}) \wedge b \in L_c$, then

$$p(./b) = p(c \vee b^{\perp}) \wedge b/b)\, p(./(c \vee b^{\perp}) \wedge b);$$

(3) if $a, b, c \in L$ and $a \vee b$, $a \vee b \vee c \in L_c$, then

$$p(a/a \vee b \vee c) = p(a/a \vee b)\, p(a \vee b/a \vee b \vee c);$$

then the function p will be called a *function of conditional probability on* L.

Proposition 2.1. *Let $B \oplus L$ be Pták's sum and h be a homomorphism from $B \oplus L$ to L. For any $m \in S(L)$, the state m_h can be described as a function of conditional probability on L.*

Proof. If h is a homomorphism from $B \oplus L$ to L, then $f_0 \circ h \leftrightarrow f_1 \circ h$. If $m \in S(L)$, then the map m_h from $B \oplus L$ into $[0,1]$, which is defined as

$$m_h\big([(a_1, b_1), \dots, (a_n, b_n)]\big) = m\big(h([(a_1, b_1), \dots, (a_n, b_n)])\big),$$

is a state of $S(B \oplus L)$. Without lost of generality we can assume that $m\big(h(f_0(a_i))\big) \neq 0$ for all $i = 1, 2, \dots, n$. Now we can calculate

$$\begin{aligned} m_h\big([(a_1, b_1), \dots, (a_n, b_n)]\big) &= m\big(h([(a_1, b_1), \dots, (a_n, b_n)])\big) \\ &= m\Big(\bigvee_{i=1}^{n} \big(h(f_0(a_i)) \wedge h(f_1(b_i))\big)\Big) \\ &= \sum_{i=1}^{n} m\Big(h\big(f_0(a_i)\big) \wedge h\big(f_1(b_i)\big)\Big) \\ &= \sum_{i=1}^{n} m\Big(h\big(f_0(a_i)\big)\Big)\, m\Big(h\big(f_0(a_i)\big) \wedge h\big(f_1(b_i)\big)\Big) \Big/ \\ &\qquad \Big/ m\Big(h\big(f_0(a_i)\big)\Big). \end{aligned}$$

It is clear that a map

$$p^m(./a) = m((.) \wedge a)/m(a)$$

if $a \in C(L)$ is a function of conditional probability and then we can write

$$m(h([(a_1,b_1),\dots,(a_n,b_n)])) = \sum_{i=1}^{n} p^m\Big(h(f_0(a_i))/1\Big)p^m\Big((h(f_1(b_i))/h(f_0(a_i))\Big).$$

If $b_i = b$ for all $i \in \{1,2,\dots,n\}$, then

$$m_h\big([(a_1,b),\dots,(a_n,b)]\big) = m\big(h([(1,b)])\big) =$$

$$= \sum_{i=1}^{n} m\Big(h(f_0(a_i))\Big)p^m\Big(h(f_1(b))/h(f_0(a_i))\Big).$$

□

COROLLARY 2.1.1. *If we assume that* $B \oplus L = C(L) \oplus L$ *and* $h(f_0(a)) = a$, $h(f_1(b)) = b$, *then for any* $[(a_1,b_1),\dots,(a_n,b_n)] \in B \oplus L$ *and for any* $m \in S(L)$

$$m_h\big([(a_1,b_1),\dots,(a_n,b_n)]\big) = \sum_{i=1}^{n} m(a_i)\, p^m(b_i/a_i).$$

If $b_i = b$ *for all* $i = 1,\dots,n$ *we get the theorem of full probability*

$$m(b) = m_h\big([1,b]\big) = \sum_{i=1}^{n} m(a_i)\, p(b/a_i).$$

3. Observables and their expectations

Let L be a σ-complete quantum logic (i.e., L is an orthomodular σ-lattice), x be an observable and m be a state on L. Then $m\colon E \to m(x(E))$ for $E \in B(\mathbb{R}_1)$ is a probability measure on $B(\mathbb{R}_1)$. The expectation of x on a state m is defined by the formula

$$m(x) = \int x \,\mathrm{d}m = \int \lambda m(\mathrm{d}\lambda)$$

if the integral on the right-hand side exists.

If $a \in L$ and x is an observable on L such that $a \leftrightarrow x$ (that is, $a \leftrightarrow x(E)$ for any $E \in B(\mathbb{R}^1)$) and $m(x)$ exists we can define

$$m(a \wedge x) = \int_a \mathrm{d}m = \int \lambda m\big(x(\mathrm{d}\lambda) \wedge a\big)\,.$$

If x_0 is an observable on L such that $x_0\big(\{0\}\big) = 1$ and and z is any other observable, where $m(z)$ exists, then for the observable $z' = (z \wedge a) \vee (x_o \wedge a^{\perp})$ (i.e., $z'(E) = \big(z(E) \wedge a\big) \vee \big(x_0(E) \wedge a^{\perp}\big)$ for all $E \in B(\mathbb{R}^1)$) we have $m(z \wedge a) = m(z')$.

Definition 3.1. [13] Let L be a q.l. and L_0 be a sublogic of L. Let x be an observable on L, m be a state on L and $a \in L \setminus \{0\}$. We will suppose that

(1) $R(x) \cup L_0$ is p.c. $[a]$;
(2) $m(a) = 1$;
(3) there exists $m(x)$.

Then by a version $E_m(x/L_0, a)$ of conditional expectation of the observable x in the state m, relativized by L_0, we understand any observable z with the properties:

(1) $z \leftrightarrow a$;
(2) $R(z) \wedge a \subset L_0 \wedge a$;
(3) for any $b \in L_0$ $\int_b x\,\mathrm{d}m = \int_b z\,\mathrm{d}m$ if the integral on the right-hand side exists.

In the following, we will assume that L will be a σ-lattice. Let B be a Boolean algebra and let $\boldsymbol{a} = (a_1, \dots, a_n)$ be an orthogonal decomposition 1 on B. Let $\boldsymbol{x} = (x_i)_{i=1}^n$ be a family of observables on L. We define $\mathbf{X}\colon B(\mathbb{R}^1) \to B \oplus L$ by $\mathbf{X} = \big[(a_1, x_1(E)), \dots, (a_n, x_n(E))\big]$, where $E \in B(\mathbb{R}^1)$. We will say that $\mathbf{X}$ is *generated by* $(\boldsymbol{a}, \boldsymbol{x}, n)$ (briefly $\mathbf{X} = \mathbf{X}(\boldsymbol{a}, \boldsymbol{x}, n)$. It is clear that $\mathbf{X}$ is an observable from $B(\mathbb{R}^1)$ into $B \oplus L$. It is known that $\sigma(\mathbf{X}) = \bigcup_{i=1}^{n} \sigma(x_i)$ ([6]).

Lemma 3.1. *Let L be a q.l. and x be an observable. If $m \in S(L)$ is such that there exist $m(x)$ then there is an observable z such that*

$$\int_b x\,\mathrm{d}m = \int_b z\,\mathrm{d}m\,,$$

for any $b \in C(L)$, where $R(z) \subset C(L)$.

Proof. It is clear that $R(x) \cup C(L)$ is p.c. $[c]$, $c \in C(L)$. Let m be a state on L and $m(c) \neq 0$. Then a map $\mu(b) = m(b \wedge c)/m(c)$ is a state on L

and moreover $\mu(c) = 1$. From this we have that there is an observable z such that $z = E_\mu\big(x/C(L),\, c\big)$ and $R(z) \wedge c \subseteq C(L) \wedge c$ with

$$\int_b x \,\mathrm{d}\mu = \int_b E_\mu\big(x/C(L),\, c\big)\,\mathrm{d}\mu\,,$$

for all $b \in C(L)$ [13]. Let us denote $z'(E) = (z(E) \wedge c) \vee \big(x_0(E) \wedge c^\perp\big)$ for any Borel set E. It is clear that z' is an observable and $\mu\big(z(E)\big) = \mu\big(z'(E)\big)$, where $E \in B(\mathbb{R}^1)$. Moreover $R(z') \subseteq C(L)$ and

$$\int_b z \,\mathrm{d}\mu = \int_b z' \,\mathrm{d}\mu\,,$$

for all $b \in C(L)$. It is clear that

$$\int_b x \,\mathrm{d}m = \int_b E_m\big(x/C(L),\, c\big)\,\mathrm{d}m = \int_b z' \,\mathrm{d}m\,.$$

$\square$

PROPOSITION 3.2. *Let $B \oplus L$ be Pták's sum. Let there be a homomorphism α from B to $C(L)$. If $\mathbf{X}(\mathbf{a}, \mathbf{x}, n)$ is an observable on $B \oplus L$ and m is a state on L such that $m(x_i)$ exists for any i, then there is an observable z on L and a map h from $B \oplus L$ to L such that, for any $b \in C(L)$,*

$$\int_{[1,b]} \mathbf{X}(\mathbf{a}, \mathbf{x}, n)\,\mathrm{d}m_h = \int_b z \,\mathrm{d}m\,,$$

$R(z) \subseteq C(L)$.

P r o o f . Let α be a homomorphism from B to $C(L)$. Then a map

$$h\big([(a_1, b_1), \dots, (a_n, b_n)]\big) = \bigvee_{i=1}^{n} \big(\alpha(a_i) \wedge b_i\big)$$

is homomorphism from $B \oplus L$ to L. Then $\mathbf{X}(a, x, n) \circ h$ is an observable on L. Now

$$m_h\big(\mathbf{X}(\mathbf{a}, \mathbf{x}, n)\big) = \int \mathbf{X}(\mathbf{a}, \mathbf{x}, n)\,\mathrm{d}m_h = \int \lambda m\Big(h\big(\mathbf{X}(\mathbf{a}, \mathbf{x}, n)(\mathrm{d}\lambda)\big)\Big)\,,$$

$$m\Big(h\big(\mathbf{X}(\boldsymbol{a},\boldsymbol{x},n)(E)\big)\Big)=m\Big(\bigvee_{i=1}^{n}\big(\alpha(a_i)\wedge x_i(E)\big)\Big)=\sum_{i=1}^{n}m\big(\alpha(a_i)\wedge x_i(E)\big)\,.$$

Then we get for any $b\in C(L)$

$$\begin{aligned}\int\limits_{[(1,b)]}\mathbf{X}\,\mathrm{d}m_h&=\int\lambda m\Big(h\big(\mathbf{X}(\boldsymbol{a},\boldsymbol{x},n)(\mathrm{d}\lambda)\wedge[(1,b)]\big)\Big)\\&=\sum_{i=1}^{n}\int\lambda m\big(\alpha(a_i)\wedge x_i(\mathrm{d}\lambda)\wedge b\big)\,.\end{aligned}$$

Without lost of generality we can assume that $m\big(\alpha(a_i)\big)\neq 0$ for all $i=1,\dots,n$ and $p^m\big((.)/\alpha(a_i)\big)=m\big((.)\wedge\alpha(a_i)\big)/m\big(\alpha(a_i)\big)$ $\big(\alpha(a_i)\in C(L)\big)$. Then there is an observable z_i such that $R(z_i)\wedge\alpha(a_i)\subseteq C(L)\wedge\alpha(a_i)$ and, for any $c\in C(L)$,

$$\begin{aligned}\int\limits_b x_i dp\big((.)/\alpha(a_i)\big)&=\int\limits_b E_{p((.)/\alpha(a_i))}\big(x_i/C(L),\alpha(a_i)\big)\,dp\,\big((.)/\alpha(a_i)\big)\\&=\int\limits_b z_i dp\big((.)/\alpha(a_i)\big)\,.\end{aligned}$$

Let us put $z_i'=(z_i\wedge a_i)\vee(x_0\wedge a_i^{\perp})$. Then

$$\begin{aligned}\sum_{i=1}^{n}\int\lambda m\big(\alpha(a_i)\wedge x_i(\mathrm{d}\lambda)\wedge b\big)&=\sum_{i=1}^{n}m\big(\alpha(a_i)\big)\int\lambda p^m\big(x_i(\mathrm{d}\lambda)\wedge b/\alpha(a_i)\big)\\&=\sum_{i=1}^{n}m\big(\alpha(a_i)\big)\int\lambda p^m\big(z_i(\mathrm{d}\lambda)\wedge b/\alpha(a_i)\big)\\&=\sum_{i=1}^{n}\int\limits_{\alpha(a_i)\wedge b}z_i\,\mathrm{d}m\\&=\sum_{i=1}^{n}m\big(z_i\wedge\alpha(a_i)\wedge b\big)=\sum_{i=1}^{n}m(z_i'\wedge b)\,.\end{aligned}$$

Since $1\in C(L)$ we have

$$m_h\big(\mathbf{X}(\boldsymbol{a},\boldsymbol{x},n)\big)=\sum_{i=1}^{n}m(z_i')=m\Big(\sum_{i=1}^{n}(z_i')\Big)\,.$$

But $\bigcup\limits_{i=1}^{n} R(z_i') \subseteq C(L)$ and there is an observable z such that $z = \sum\limits_{i=1}^{n} z_i'$. Then

$$\int\limits_{[(1,b)]} \mathbf{X}(a, x, n)\, \mathrm{d}m_h = \int\limits_{b} z \,\mathrm{d}m\,.$$

□

EXAMPLE 3.1. Let $(\Omega_1, \mathcal{F}_1, m_1)$ and $(\Omega_2, \mathcal{F}_2, m)_1$ be two probability spaces such that $\Omega_1 = \Omega_2 = \{1,2,3,4,5,6\}$, $\mathcal{F}_1$ have atoms $\{\{1\},\{2\},\{3,4,5,6\}\}$ and $\mathcal{F}_2$ have atoms $\{\{3\},\{1\},\{1,4,5,6\}\}$. Let $m_1(\{1\}) = 1/4$, $m_1(\{3\}) = 1/2$, $m_1(\{2,4,5,6\}) = 1/4$, $m_2(\{1\}) = 1/4$, $m_2(\{2\}) = 1/2$, $m_2(\{3,4,5,6\}) = 1/4$. It is clear that $L = \mathcal{F}_1 \cup \mathcal{F}_2$ is a q.l. (Fig. 1 (Greechie diagram) [14]).

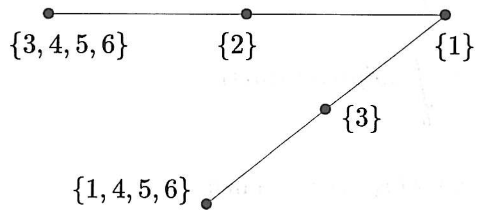

FIG. 1.

It is clear that a function m is the state on L if $m(a) = m_i(a)$ for any $a \in \mathcal{F}_i$ $(i = 1,2)$.

Let us define maps x, y from $B(\mathbb{R}^1)$ to L as follow $x(\{1\}) = \{1\}$, $x(\{3\}) = \{3\}$, $x(\{2\}) = \{2,4,5,6\}$, $y(\{3\}) = \{1\}$, $y(\{2\}) = \{2\}$, $y(\{5\}) = \{3,4,5,6\}$. It is clear that x, y are observable on L and $\sigma(x) = \{1,2,3\}$, $\sigma(y) = \{2,3,5\}$. We can calculate expectations of these observables in the state m:

$$m(x) = 1m(\{1\}) + 3m(\{3\}) + 2m(\{2,4,5,6\}) = 9/4\,,$$
$$m(y) = 3m(\{1\}) + 2m(\{2\}) + 5m(\{3,4,5,6\}) = 3\,.$$

The set $C(L) = \{\{1\},\{2,3,4,5,6\},\{1,2,3,4,5,6\},\emptyset\}$ is the center of L. The set $C(L) \oplus L$ is a Pták's sum and $\mathbf{X} = [(\{1\}, x),\ (\{2,3,4,5,6\}, y)]$ is the observable on $C(L) \oplus L$. A homomorphism h from $C(L) \oplus L$ to L is defined as follows:

$$h\big([(a,b),(a^{\perp},c)]\big) = (a \wedge b) \vee (a^{\perp} \wedge c)\,,$$

where $a \in C(L),\ \ b, c \in L$. We get

$$\begin{aligned}
\mathbf{X}(\{1\}) &= \big[(\{1\},(\{1\}),(\{2,3,4,5,6\},\emptyset)\big]\,,\\
\mathbf{X}(\{2\}) &= \big[(\{1\},(\{2\}),(\{2,3,4,5,6\},\{2\})\big]\,,\\
\mathbf{X}(\{3\}) &= \big[(\{1\},(\{3\}),(\{2,3,4,5,6\},\{1\})\big]\,,\\
\mathbf{X}(\{5\}) &= \big[(\{1\},\emptyset),(\{2,3,4,5,6\},\{3,4,5,6\})\big]\,.
\end{aligned}$$

Then

$$\begin{aligned}
&m_h(\mathbf{X}) = m(\{1\}) + 2m(\{2\}) + 5m(\{3,4,5,6\}) = 5/2\,,\\
&m_h\big(\mathbf{X} \wedge [(\{1,2,3,4,5,6\}),\ (\{1\})]\big) = 1/4\,,\\
&m_h\big(\mathbf{X} \wedge [(\{1,2,3,4,5,6\}),(\{2,3,4,5,6\})]\big) = 9/4\,.
\end{aligned}$$

Let z be an observable such that $R(z) = C(L)$ and $m(z) = 5/2$. Then we get $m(z) = t_1 m(\{1\}) + t_2 m(\{1\}^{\perp}) = t_1/4 + t_2 3/4$. From this we have $\sigma(z) = \{10 - 3t_2, t_2\}$, and

$$\begin{aligned}
m_h\big(z \wedge \{1\}\big) &= t_1/4\,, & \qquad m_h\big(z \wedge \{2,3,4,5,6\}\big) &= 3t_2/4\,,\\
t_1 &= 3\,, & t_2 &= 1\,.
\end{aligned}$$

It is clear that the observable z is a version of conditional expectation $E_m\big(h \circ \mathbf{X}/C(L),\, 1\big)$.

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