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EMBEDDING OF ORTHOPOSETS INTO ORTHOCOMPLETE POSETS

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ABSTRACT. In this paper we deal with the embedding of an orthoposet into an orthocomplete poset, i.e., into an orthoposet in what there exists the supremum of any set of pairwise orthogonal elements. This embedding is constructed using a restriction of the well-known MacNeille completion. We show a necessary and sufficient condition for preserving the property of orthomodularity. There is given an example of an orthomodular poset, which possesses an orthomodular orthocompletion, but its MacNeille completion is not orthomodular. Besides, we give an example of an orthomodular poset, whose orthocompletion is not orthomodular.

1. Preliminaries

Let $(L, \leqq)$ be a partially ordered set (poset). For any $Z \subset L$ denote Z^* (Z^+) the set of all upper (lower) bounds of L. It is known that L can be embedded into a complete lattice. This complete lattice is the *MacNeille completion* $MC(L) = \{Z^{*+};\ Z \subset L\}$ (see [7]). For any system $Z_t^{*+} \in MC(L)$,

$$\bigvee Z_t^{*+} = \Big(\bigcup Z_t\Big)^{*+},$$

$$\bigwedge Z_t^{*+} = \bigcap Z_t^{*+}.$$

The embedding of L into $MC(L)$ is the mapping $\varphi\colon L \to MC(L)$, $\varphi(a) = (a\rangle = \{b \in L;\ b \leqq a\}$. The MacNeille completion preserves all joins existing in L. However, it need not save, in general, some other properties of L, if they exist.

DEFINITION 1.1. The *orthoposet* is a triple $(L, \leqq, {}')$, where L is a set partially ordered by $\leqq$, possessing the smallest element 0 and the largest element 1, and $'\colon L \to L$ is the operation of *orthocomplementation* with properties:

(i) $(a')' = a$,
(ii) $a \leqq b$ implies $b' \leqq a'$,
(iii) $a \vee a' = 1$,
(iv) $a \leq b'$ implies $a \vee b \in L$.

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Let us define the *MacNeille orthocompletion* of L

$$MOC(L) = \bigcap_{C \in \mathfrak{S}} C\,.$$

$MOC(L)$ is the orthocomplete poset and $\varphi\colon L \to MOC(L)$, $\varphi(a) = (a\rangle$, is the embedding of L into $MOC(L)$. $MOC(L)$ can be defined also by the transfinite induction. Let us define

$$C_1 = \{Z^{*+};\ Z \text{ is orthogonal subset of } L\} \cup \{\{0\}\}\,.$$

For every ordinal number $\alpha < \Gamma$, where Γ is an ordinal number of the potence set of $MC(L)$ (by some well-ordering), let us define

$$C_\alpha = \Big\{\big(\bigcup Z_t\big)^{*+};\ Z_t^{*+} \text{ are pairwise orthogonal, } Z_t^{*+} \in C_\beta \text{ for some ordinal } \beta < \alpha\Big\}$$
$$\cup \big\{Z^{*+\perp};\ Z^{*+} \in C_\beta \ \text{ for some } \ \beta < \alpha\big\}\,.$$

Then $C_\Gamma = MOC(L)$. Of course, using the transfinite induction, it can be easily shown that $C_\alpha \subset MOC(L)$ for every ordinal $\alpha \leqq \Gamma$. $C_\Gamma \subsetneqq MOC(L)$ would imply the existence of a transfinite sequence $\{Z_\alpha^{*+}\}_{\alpha \leqq \Gamma}$ of elements of $MC(L)$ with the property $Z_\alpha^{*+} \in C_\alpha$, $Z_\alpha^{*+} \notin \bigcup_{\beta<\alpha} C_\beta$, $\alpha \leqq \Gamma$. This would imply $\operatorname{card}\{Z_\alpha^{*+};\ \alpha \leqq \Gamma\} = 2^{\operatorname{card} MC(L)}$, what is a contradiction.
L is join-dense in $MC(L)$, thus, of course, it is join-dense in $MOC(L)$. Thus, the embedding $\varphi\colon L \to MOC(L)$ preserves all joins existing in L.

PROBLEM. Is $MOC(L)$ the smallest orthocompletion of L in such sense that any other embedding $\varphi'\colon L \to P$, where P is orthocomplete, can be written as $\varphi' = \psi \circ \varphi$, where ψ is an embedding of $MOC(L)$ into P (so as $MC(L)$ is the smallest completion of L in the sense of the embedding of L into a complete lattice)?

3. Orthomodularity of the orthocompletion

The embedding of L into an orthocomplete poset instead of a complete lattice could be useful especially in such cases, when it saves the orthomodularity. The poset L given in the mentioned example in [5] is orthomodular and orthocomplete, thus it is isomorphic to $MOC(L)$. So, it is a trivial example of a poset with $MOC(L)$ orthomodular and $MC(L)$ non-orthomodular. In the following example, we have L orthomodular and non-orthocomplete, $MOC(L)$ orthomodular and $MC(L)$ non-orthomodular.

EXAMPLE 3.1. Let $X = \langle 0, 1\rangle$, $Y = \{1,2,3,4,5,6\}$. Let L be the concrete logic of subsets of $X \times Y$ of the form $A \times Y$, where A is a borel subset of X, or $X \times B$, where $B \subset Y$, card B is even. Then $MOC(L)$ is isomorphic with orthogonal and $Z \cup W$ be maximal orthogonal. By Lemma 3.3, $\bigvee_{z \in Z} \varphi(z) = \left(\bigvee_{w \in W} \varphi(w)\right)'$. If $a \in W^{\perp}$, then $\varphi(a) \perp \varphi(w)$ for $w \in W$. This implies $\varphi(a) \leqq \bigvee_{z \in Z} \varphi(z)$. Hence, $\varphi(a) \perp \varphi(t)$ for every $t \in W_1$. So, $a \in W_1^{\perp}$. The property (iii) is verified.

(iii) $\Rightarrow$ (iv) Let (iii) be satisfied. We shall prove that this implies the orthomodularity of $MOC(L)$. At first we prove that (iii) implies $MOC(L) = C_1$. Let Z, W be nonempty, $Z \cap W = \emptyset$ and $Z \cup W$ be a maximal orthogonal subset of L. We shall prove that $W^{*+} = Z^{*+\perp}$. Obviously, $W^{*+} \subset Z^{*+\perp}$. If $0 \neq a \in Z^{*+\perp}$, then $Z \cup \{a\}$ is an orthogonal set. By (iii), $W^{\perp} \subset \{a\}^{\perp}$. Let $s \in W^*$. Then $s' \in W^{\perp} \subset \{a\}^{\perp}$, thus $s' \perp a$. Hence, $a \leqq s$ for every $s \in W^*$. So, $a \in W^{*+}$. We have $W^{*+} = Z^{*+\perp}$. The immediate consequence is $MOC(L) = C_1$.

Now, we prove the orthomodularity of $MOC(L)$. Let Z^{*+}, $W^{*+} \in MOC(L)$, where Z, W are orthogonal subsets of L, $\{0\} \neq Z^{*+} \subsetneqq W^{*+} \neq L$. Let W_1 be an orthogonal set such that $W_1 \cap W = \emptyset$ and $W_1 \cup W$ is maximal orthogonal. Then $W_1 \cup Z$ is also orthogonal. Let S be an orthogonal set such that $S \cap (W_1 \cup Z) = \emptyset$ and $Z \cup S \cup W_1$ is maximal orthogonal. Then we obtain

$$(Z \cup S)^{*+} = W_1^{*+\perp} = W^{*+},$$
$$S^{*+} = (Z \cup W_1)^{*+\perp}.$$

We have $(Z \cup S)^{*+} = Z^{*+} \vee S^{*+}$ and $(Z \cup W_1)^{*+\perp} = (Z^{*+} \vee W_1^{*+})^{\perp} = Z^{*+\perp} \wedge W_1^{*+\perp} = Z^{*+\perp} \wedge W^{*+}$. So, we have obtained the orthomodular law:

$$W^{*+} = Z^{*+} \vee (Z^{*+\perp} \wedge W^{*+}).$$

(iv) $\Rightarrow$ (i). Obvious.

Theorem is proved. □

We give an example of an orthomodular poset L, which cannot be embedded into any orthomodular orthocomplete poset in such a way, that orthogonal all joins existing in L will be saved.

EXAMPLE 3.5. Let T, X, X_t, $t \in T$, be sets with properties: card $T = c$ (c is the cardinal of real line), card $X_t = c$, $X = \bigcup_{t \in T} X_t$, X_t are pairwise disjoint. Let L be the collection of all countable subsets A of X such, that card $A \cap X_t \leqq 1$ for every $t \in T$, and of all set-theoretical complements of these sets. Let us define the partial ordering $\leqq$ on L as follows:

If A, B are countable, we put $A \leqq B$, if $A \subset B$.

If A is countable and B is not, then $A \leqq B$, if for every $t \in T$, $A \cap X_t \neq \emptyset$ implies $X_t \subset B$.

If A, B are not countable, then we put $A \leqq B$, if $A \subset B$.