Tatra Mountains Math. Publ. **3** (1993), 161–166

TATRA MOUNTAINS
Mathematical Publications

A CHARACTERIZATION OF TRIANGULAR NORM BASED TRIBES

MIRKO NAVARA

ABSTRACT. We study collections of fuzzy sets which are closed under complementation and under countable products (interpreted as "fuzzy intersections"). We characterize them as certain spaces of functions measurable with respect to the σ-algebra of crisp elements. This solves the problem stated in [4] and [5] and enables a generalization of results of [1, 2, 3, 4].

1. Preliminaries

Fuzzy probability models generalize classical Kolmogorov probability theory. We start with a nonempty set X and a collection $F \subset [0,1]^X$ of fuzzy sets equipped with some pointwise operations. The complement of $f \in F$ is usually defined as $1-f$. The fuzzy intersection is defined by the pointwise application of a commutative associative operation $T : [0,1]^2 \to [0,1]$ called a triangular norm. There are many kinds of triangular norms; here we restrict our attention to the triangular norm T_1, which is the ordinary multiplication. It may serve as a "typical" representant of a larger class of triangular norms – see [5] for a more general overview. We recall here only those definitions which will be necessary in the sequel.

DEFINITION 1.1 ([2]). Let X be a nonempty set. A T_1-*tribe* on X is a collection $F \subset [0,1]^X$ such that

1. $0 \in F$,
2. $f \in F \Longrightarrow 1-f \in F$,
3. $\{f_n\}_{n\in\mathbb{N}} \subset F \Longrightarrow \prod_{n\in\mathbb{N}} f_n \in F$

(0, 1 denote the constant functions on X).

AMS Subject Classification (1991): 28E10, 46S10, 28A20.
Key words: fuzzy σ-algebra, generated tribe, t-norm, T-tribe, measurable function.

The author gratefully acknowledges the support of the grant no. 201/93/0953 of Grant Agency of the Czech Republic.

Throughout this paper, $F \subset [0,1]^X$ is a T_1-tribe. A subset $G \subset F$ is a *sub-T_1-tribe* (a *subtribe*, for short) of F if it is a T_1-tribe with respect to the operations inherited from F. For $Y \subset X$, we denote by $F|Y$ the restriction $\{f|Y\colon f \in F\}$ (which is a T_1-tribe as well). The collection $C(F) = F \cap \{0,1\}^X$ is the σ-algebra of *crisp elements* of F. The mapping $i : f \mapsto f^{-1}(\{1\})$ is an isomorphism of $C(F)$ and $i\big(C(F)\big) \subset \exp X$. If we say that an $f \in [0,1]^X$ is *measurable with respect to crisp elements*, we mean that f is measurable with respect to $i\big(C(F)\big)$. For $f \in [0,1]^X$, we define the *crisp domain* $D_C(f) = \big\{x \in X\colon f(x) \in \{0,1\}\big\}$ and the *fuzzy domain* $D_F(f) = \big\{x \in X\colon f(x) \in (0,1)\big\}$.

DEFINITION 1.2 ([2]). A collection $G \subset [0,1]^X$ is called

- a *generated tribe* if there is a σ-algebra $C \subset \exp X$ such that $G = \big\{f \in [0,1]^X\colon f$ is C-measurable$\big\}$,
- a *semigenerated tribe* if there is a σ-algebra $C \subset \exp X$ and $Y \subset X$ such that $G = \big\{f \in [0,1]^X\colon f$ is C-measurable, $f|Y$ is crisp$\big\}$,
- a *weakly generated tribe* if there is a σ-algebra $C \subset \exp X$ and a σ-ideal Δ in C such that $G = \big\{f \in [0,1]^X\colon f$ is C-measurable, $D_F(f) \in \Delta\big\}$.

A weakly generated tribe corresponding to a σ-ideal Δ is semigenerated (resp. generated) iff the ideal Δ is principal (resp. non-proper).

Remark 1.3. The notion of generated tribe is commonly used, semigenerated tribes are introduced in [4] and the notion of weakly generated tribe is new.

PROPOSITION 1.4. *Every weakly generated tribe is a T_1-tribe.*

The proof requires only a routine verification. □

2. Main result

Many results (see [1]–[5]) were obtained for generated or semigenerated tribes and it is desirable to find their generalizations for all triangular norm based tribes (in particular, for T_1-tribes). It was not clear to which extent the general case might be different.

As the main result of this paper we shall prove that T_1-tribes coincide with weakly generated tribes (Theorem 2.3). We need to recall some preceding partial results towards this direction.

THEOREM 2.1 ([1]). *A T_1-tribe is closed under countable (pointwise) suprema and infima.*

THEOREM 2.2 ([1, 4]). *If a T_1-tribe contains a constant function different from* 0 *and* 1, *it is a generated tribe.*

Our characterization is the following:

THEOREM 2.3. *Every T_1-tribe is a weakly generated tribe (and vice versa).*

The proof of Theorem 2.3 will require several lemmas. First of all we shall describe all possible operations that can be applied pointwise to an element of a T_1-tribe. For this purpose we introduce the notion of admissible function.

DEFINITION 2.4. The set A of *admissible functions* is the smallest collection of functions $p : [0,1] \to [0,1]$ such that

1. the identity $\mathrm{id} \in A$,
2. the constant function $0 \in A$,
3. $p \in A \Longrightarrow 1 - p \in A$,
4. $p, q \in A \Longrightarrow p \circ q \in A$,
5. $\{p_n\}_{n\in\mathbb{N}} \subset A \Longrightarrow \prod_{n\in\mathbb{N}} p_n \in A$.

Notice that the set of admissible functions is a T_1-tribe and, according to Theorem 2.1, it is a σ-lattice.

LEMMA 2.5. *If $f \in F$ and $p \in A$, then $p \circ f \in F$.*

Proof. The same sequence of operations which generated p from id can be applied pointwise to f and results in $p \circ f$. □

Lemma 2.5 allows a description of some procedures applied to elements of F without reference to the points of X. Thus our problem is transferred to the investigation of admissible functions.

For a set M, we denote by χ_M its characteristic function. (The domain of this function will be clear from the context.)

COROLLARY 2.6. *If $\chi_B \in A$ for some Borel set B, then $\chi_{f^{-1}(B)} = \chi_B \circ f \in C(F)$.*

For each $k, n \in \mathbb{N}$ we define an increasing admissible function $p_{k,n}(t) = \left(1 - (1-t)^k\right)^n$. These functions will play an important role in several steps of the following proofs.

LEMMA 2.7. *For each $a, b, t \in (0,1)$, $a < b$, there are $k, n \in \mathbb{N}$ such that $p_{k,n}(t) \in [a,b]$.*

Proof. Notice that $\{p_{k,n}(t)\}_{n\in\mathbb{N}}$ is a geometric sequence with the quotient $1-(1-t)^k < 1$. For k such that $(1-t)^k \le b - a$, at least one element of this sequence belongs to the interval $[a,b]$. □

LEMMA 2.8. *For each $a, b, r \in (0,1)$, $a < b$, there are $k, n \in \mathbb{N}$ such that*

$$p_{k,n}(a) \le r < p_{k,n}(b).$$

P r o o f . The required inequality can be rewritten to an equivalent form

$$1 - a \ge \sqrt[k]{1 - \sqrt[n]{r}} > 1 - b$$

which is satisfied for some $k, n \in \mathbb{N}$. □

LEMMA 2.9. *Let $\chi_{[0,r]} \in A$ for some $r \in (0,1)$. Then $\chi_{[0,a]} \in A$ for all $a \in [0,1]$.*

P r o o f . The case $a \in \{0,1\}$ is trivial; suppose that $a \in (0,1)$. We define a function $p = \bigwedge\limits_{k,n} \chi_{[0,r]} \circ p_{k,n}$, where the infimum is taken over all k, n such that $p_{k,n}(a) \le r$. The function p is admissible and it is the characteristic function of some interval $[0,b]$, $b \ge a$. If $b > a$, Lemma 2.8 gives a contradiction, hence $p = \chi_{[0,a]}$. □

LEMMA 2.10. *There is an $r \in (0,1)$ such that $\chi_{[0,r]}$ is admissible.*

P r o o f . We take an admissible function $q_0(t) = p_{2,2}(t) = \big(1 - (1-t)^2\big)^2$. The equation $q_0(t) = t$ has exactly one root in $(0, 1/2)$, namely $(3 - \sqrt{5})/2$; we take this root for r. The function q_0 satisfies the following properties:

- $q_0(t) = t$ for $t \in \{0, r, 1\}$,
- $q_0(t) < t$ for $t \in (0, r)$,
- $q_0(t) > t$ for $t \in (r, 1)$.

We define a sequence $\{q_n\}_{n\in\mathbb{N}} \subset A$ recursively: $q_n(t) = q_0\big(q_{n-1}(t)\big)$. We obtain

$$\lim_{n\to\infty} q_n(t) = 1 \quad \text{for all} \quad t \in (r, 1].$$

The function $q = \bigvee\limits_{n\in\mathbb{N}} q_n \in A$ satisfies the following properties:

- $q(t) = 1$ for $t \in (r, 1]$,
- $q(t) \le t$ for $t \in [0, r]$.

As $r < 1 - r$, the admissible function $\overline{q}(t) = q\big(1 - q(t)\big)$ is the characteristic function of $[0, r]$. □

P r o o f o f T h e o r e m 2.3. According to Lemmas 2.10 and 2.9, all characteristic functions of intervals $[0,s]$, $s \in [0,1]$, are admissible. A standard argument allows to generalize this observation to all Borel subsets of $[0,1]$. Let $C = i(C(F)) \subset \exp X$ be the σ-algebra corresponding to the σ-algebra of crisp elements of F. Corollary 2.6 implies that all $f \in F$ are C-measurable.

We define $\Delta = \{D_F(f) \colon f \in F\}$. Let us prove that Δ is a σ-ideal in C. First, $\emptyset = D_F(0) \in \Delta$. Second, if $M \in \Delta$, $K \in C$ and $K \subset M$, then $\chi_K \in F$. There is an $f \in F$ such that $M = D_F(f)$. We obtain $K = D_F(f \wedge \chi_K) \in \Delta$. Third, let $\{M_n\}_{n\in\mathbb{N}} \subset \Delta$. There is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset F$ such that $M_n = D_F(f_n)$ for all $n \in \mathbb{N}$. We define an admissible function $u = \bigvee\limits_{k,n} (p_{k,n} \wedge (1-p_{k,n}))$ (the supremum is taken over all $k,n \in \mathbb{N}$). It is clear that $u(0) = u(1) = 0$ and $u(t) \leq 1/2$ for all $t \in (0,1)$. According to Lemma 2.7, for each $\varepsilon > 0$ and each $t \in (0,1)$ there exist $k,n \in \mathbb{N}$ such that $p_{k,n}(t) \in [1/2 - \varepsilon, 1/2]$. This implies that $u(t) = 1/2$ for all $t \in (0,1)$. The function $g = \bigvee\limits_{n\in\mathbb{N}} u \circ f_n$ belongs to F (Lemma 2.5) and satisfies $D_F(g) = g^{-1}(\{1/2\}) = \bigcup\limits_{n\in\mathbb{N}} D_F(f_n) = \bigcup\limits_{n\in\mathbb{N}} M_n$, hence $\bigcup\limits_{n\in\mathbb{N}} M_n \in \Delta$ and Δ is a σ-ideal. Moreover, for each $f \in F$ the restriction $F|D_F(f)$ contains the constant function $1/2 = (u \circ f)|D_F(f)$ and hence (Th. 2.2) also all C-measurable functions on $D_F(f)$. Thus F contains all C-measurable functions whose fuzzy domains belong to Δ, and the proof of Theorem 2.3 is complete. □

3. Corollaries and examples

Our characterization leads to weakly generated tribes which are a proper generalization of generated tribes. In order to clarify the similarities and differences, we add some other ideas. The corollaries were obtained as a by-product of the proof of the main theorem.

Corollary 3.1. *Every weakly generated tribe F is a subtribe of a generated tribe G with the same crisp elements, i.e., $C(F) = C(G)$.*

Corollary 3.2. *Each element f of a weakly generated tribe F is contained in a semigenerated subtribe of F.*

The last corollary shows that a weakly generated tribe contains a large "three-valued sublogic" (which, of course, is not a subtribe).

Corollary 3.3. *Let F be a weakly generated tribe and let $f \in F$. We* *define* $\Xi f \in \{0, 1/2, 1\}^X$ *by the following rules:*

$$\Xi f(x) = \begin{cases} 0, & \text{for } f(x) = 0, \\ 1, & \text{for } f(x) = 1, \\ 1/2, & \text{for } f(x) \in (0,1). \end{cases}$$

Then $\Xi f \in F$.

Proof. Using the function u from the final part of the proof of Theorem 2.3, we define an admissible function $w = u \vee \chi_{\{1\}}$. We obtain $w \circ f = \Xi f$ and Lemma 2.5 completes the proof. □

The following examples show that weakly generated tribes are a proper generalization of generated tribes.

EXAMPLE 3.4. Let X be an uncountable set. We define $F = \{f \in [0,1]^X :$ $D_F(f)$ is countable$\}$, $G = \{f \in [0,1]^X :$ one of the sets $X \setminus f^{-1}(\{0\})$, $X \setminus f^{-1}(\{1\})$ is countable$\}$. Then F, G are weakly generated tribes which are not semigenerated.

EXAMPLE 3.5. Let (X, C, P) be a (classical) probability space. Let $F = \{f \in [0,1]^X : f$ is C-measurable, $P(D_F(f)) = 0\}$ (i.e., the elements of F are crisp P-almost everywhere). Then F is a weakly generated tribe.

REFERENCES

[1] BUTNARIU, D—KLEMENT, E. P.: *Triangular norm-based measures and their Markov-kernel representation*, J. Math. Anal. Appl. **162** (1991), 111–143.

[2] KLEMENT, E. P.: *Construction of fuzzy σ-algebras using triangular norms*, J. Math. Anal. Appl. **85** (1982), 543–565.

[3] KLEMENT, E. P.—LOWEN, R.—SCHWYHLA, W.: *Fuzzy probability measures*, Fuzzy Sets and Systems **5** (1981), 21–30.

[4] MESIAR, R.: *Fundamental triangular norm based tribes and measures*, J. Math. Anal. Appl. **177** (1993), 633–640.

[5] MESIAR, R.: *Fuzzy sets and probability theory*, Tatra Mountains Math. Publ. **1** (1992), 105–123.

Received February 4, 1993

Department of Mathematics
Faculty of Electrical Engineering
Czech Technical University
CZ–166 27 Praha 6
CZECH REPUBLIC
E-mail: navara@math.feld.cvut.cz