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ORDER AND ORTHOGONALITY RELATIONS IN RINGS AND ALGEBRAS

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ABSTRACT. In this paper we consider well-known and new order relations in rings with involution, alternative rings and Jordan algebras. To each of them is associated an orthogonality relation and the notion of orthomodular group is introduced in order to collect their common properties related to the orthomodular law. Commutativity of two elements (in the meaning of orthomodular lattice theory) is characterized and some particular properties of idempotents and projections are proved.

Introduction

In a large class of rings (not necessarily associative), the binary relation $p \leq q$ if and only if $p = pq = qp$ is an order relation in the set of idempotents and, for rings with involution, $p = pq$ orders the set of projections. In such rings, two idempotents p and q are said to be orthogonal if $qp = pq = 0$ and these relations often lead to structures related to orthomodular lattices. Natural problems are then, how to extend these order and orthogonality relations to the set of all elements of the rings, to find some common properties of the different extensions and to link order relations and orthogonality relations. The aim of this paper is to provide answers to these problems in such a way that the extensions preserve the orthomodular properties of the initial ordering.

The paper is organized as follows. In Section 1, we give definitions and elementary properties of order relations in classes of rings with involution, alternative rings, Jordan rings and algebras. Section 2 is devoted to the introduction of orthomodular groups in order to gather the common properties of the different order relations. In Section 3, we make explicit orthogonality relations which are associated to each order relation introduced in the first section. Commutativity of two elements (in the meaning of orthomodular lattice theory) for the previous

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order relations is characterized in Section 4. In the final section we have collected miscellaneous properties of idempotents and projections.

For notions concerning orthomodular posets and lattices, let us refer to [18]. Basic information on Jordan rings and algebras may be found in [11] and for rings with involution [2] is a standard reference.

We assume that all the fields considered are of characteristic different from 2.

1. Definitions of the order relations

The aim of this section is to give definitions and elementary properties of order relations in classes of $\star$-rings, alternative rings and Jordan rings and algebras.

1.1 $\star$-rings and $\star$-algebras.

A *$\star$-semi-group* S is a semigroup equipped with an involution $x \mapsto x^\star$ such that

$$(xy)^\star = y^\star x^\star .$$

An element p of a $\star$-semigroup is called a *projection* if $p = p^2 = p^\star$.

A ring with involution A is a *$\star$-ring* if

$$(x+y)^\star = x^\star + y^\star$$

and if the multiplicative semigroup of A is a $\star$-semi-group. When A is also an algebra, over a field with involution $\lambda \mapsto \lambda^\star$ (the identity involution is allowed), we assume further that

$$(\lambda x)^\star = \lambda^\star x^\star$$

and call A a *$\star$-algebra.* For example, if H is a Hilbert space, then the algebra $\mathcal{L}(H)$ of all bounded linear operators in H, with the adjoint operation as involution, is a $\star$-algebra; so are all $\star$-subalgebras of $\mathcal{L}(H)$ and, in particular, von Neumann algebras and $C^\star$-algebras.

In a $\star$-semigroup S, M. P. Drazin [7] introduced a binary relation $\leq$ by:

$$a \leq b \iff aa^\star = ba^\star \quad \text{and} \quad a^\star a = a^\star b \tag{1}$$

and proved that this relation is an order relation, called the *$\star$-order*, if S satisfies the following condition:

$$a^\star a = a^\star b = b^\star a = b^\star b \Rightarrow a = b .$$

In a $\star$-ring, the binary relation (1) is an order relation if and only if the involution is *proper*; that means $a^\star a = 0$ implies $a = 0$.

Note that the involution in a $C^{\star}$-algebras is proper since $\|xx^{\star}\| = \|x\|^2$. The involution is also proper in every Rickart $\star$-ring. Recall that a Rickart $\star$-ring A in a $\star$-ring in which the right-annihilator of every element a in A, $\{x \in A \mid ax = 0\}$, is a principal right ideal generated by a projection noted a'. In such a ring, $(a^{\star}a)' = a'$ and so $a^{\star}a = 0$ implies a $= 0$.

1.2 Alternative rings and alternative algebras.

An *alternative ring* is a nonassociative ring satisfying the two equations:

$$x^2y = x(xy)\,, \qquad yx^2 = (yx)x\,.$$

Alternative rings can also be characterized by the fact that every subring generated by two elements is associative. In view of this result, the following, in an alternative ring without non-zero nilpotent element, can be shown to be exactly the same as the associative case:

$$xy = 0 \quad \text{if and only if} \quad yx = 0\,,$$

and

$$\text{if} \quad xy = x^2 \quad \text{then} \quad xy = yx\,.$$

The Cayley-Dickson algebras are an important class of alternative rings which are not associative.

The usual order relation in Boolean rings, $a \leq b$ if and only if $a = ab$, is a particular case of the relation $a \leq b$ if and only if $a^2 = ab$ which is an order relation in any ring without non-zero nilpotent elements [5]. In [17], this result is extended to alternative rings and the following proposition is proved:

PROPOSITION 1. *Let A be an alternative ring. The binary relation defined by $x \leq y$ if and only if $xy = x^2$ is an order relation on A if and only if A has no non-zero nilpotent element.*

1.3 Jordan rings and Jordan algebras.

A *Jordan ring* is a nonassociative ring satisfying the two equations:

$$xy = yx\,, \qquad x(yx^2) = (xy)x^2\,.$$

If A is an algebra, then the formula

$$a \circ b = \frac{ab + ba}{2}\,.$$

defines a new product $\circ$ on A. Let A^J be the algebra having the same underlying vector space as A but with the product $\circ$. If A is an associative algebra, then $\circ$ is called the *special Jordan product* on A. Any Jordan algebra isomorphic to a Jordan subalgebra of A^J, for an associative algebra A, is said to be a *special Jordan algebra.* A Jordan algebra which is not special is called an *exceptional Jordan algebra.* A typical example of an exceptional Jordan algebra is the hermitian part, equipped with the special Jordan product, of the set of 3×3 matrices with coefficients in the ring of octonions (or Cayley numbers). This algebra will be denoted by M_8^3.

Calculations in Jordan algebras used the two following theorems:

- Any Jordan algebra generated by two elements (and 1, if unital) is special (Shirshov–Cohn theorem);
- Any polynomial identity in three variables, with degree at most 1 in one variable, and which holds in all special Jordan algebras, holds in all Jordan algebras (Macdonald's theorem).

We shall need to use the operator U_a defined by $U_a(x) = 2a(ax) - a^2x$. Notice that, in a special Jordan algebra, $U_a(x) = axa$ if, here, xy denotes the associative product.

The previous order relation introduced in alternative rings is not always transitive in a Jordan algebra. Recall an example of [9]. Let Q be the ring of quaternions with usual notations and consider Q^J. One easily checks that $(i+j)^2 = (i+j) \circ (2i)$ and $(2i)^2 = 2i \circ (2i+j)$ but $(i+j)^2 \neq (i+j) \circ (2i+j)$. Thus, a different definition is necessary.

It is shown in [9] that the binary relation $\leq$ defined by

$$a \leq b \quad \text{if and only if} \quad ab = a^2,\ a^2b = ab^2 = a^3$$

is an order relation in any Jordan ring A such that

- $2a = 0$ implies $a = 0$,
- There exists in A no non-zero nilpotent element,
- The following condition holds:

$$(x, x, y) = 0 \quad \text{implies} \quad (xy, x, y) = 0, \tag{P}$$

where (x, y, z) is the associator of x, y, z defined by $(x, y, z) = (xy)z - x(yz)$.

The following lemma proves that in the definition of the order relation a simplification is possible for Jordan algebras.

LEMMA 1. *Let a and b be two elements of a Jordan algebra A without non-zero nilpotent elements.*

1. *$ab = 0$ and $a^2b = 0$ imply $ab^2 = 0$,*
2. *$ab = a^2$ and $a^2b = a^3$ imply $ab^2 = a^3$.*

P r o o f. 1) Assume $ab = 0$ and $a^2 b = 0$. Then we have $U_a(b) = 0$, $U_b(a) = -ab^2$ and $U_b(a^2) = -b^2 a^2$. By choosing $x = b$ and $y = a$ in the identity

$$4(xy)^2 = 2y\, U_x(y) + U_x(y^2) + U_y(x^2)\,, \tag{2}$$

which is satisfied in any Jordan algebra (use Macdonald' s theorem for a proof), we obtain $a^2 b^2 = 0$. Now $x = a$ and $y = b$ give $2a(ab^2) = 0$ and so $U_a(b^2) = 0$. The identity

$$U_x(y)^2 = U_x U_y(x^2) \tag{3}$$

implies $U_b(a) = 0$ and thus $ab^2 = 0$.

2) If $ab = a^2$ and $a^2 b = a^3$, then $a(b - a) = a^2(b - a) = 0$ and 1) implies $a(b - a)^2 = 0$ and therefore $ab^2 = a^3$. □

This lemma shows that, in a Jordan algebra satisfying (P) and without non-zero nilpotent element we have an ordering defined by:

$$a \le b \quad \text{if and only if} \quad ab = a^2, \quad a^2 b = a^3\,.$$

The binary relation $ab = a^2$, $a^2 b = a^3$ will be called the *J-relation* and the *J-order* when it is an order relation. Note that the property (P) is only used in [9] for proving transitivity of the J–relation.

The J–relation naturally suggests the following questions:

Q_1: Is condition (P) necessary for the transitivity of the J–relation? In other words, is the J–relation an order relation in every Jordan algebra without non-zero nilpotent element [9]?

Q_2: Does (P) hold in every Jordan algebra?

Q_3: Are the two conditions in the definition of the J–relation independent?

By [1], one can construct an example which shows that $ab = a^2$ does not imply $a^2 b = a^3$ in a Jordan algebra without any non-zero nilpotent element. Consider, in the exceptional Jordan algebra M_8^3, the two matrices

$$a = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & i-k & 0 \\ k-i & 0 & k-i \\ 0 & i-k & 0 \end{pmatrix},$$

where $ik = -j$, $ki = j$ and $i^2 = -1$, we have $a \circ b = a^2$ and $a^2 \circ b \ne a^3$. On the other hand, does $a^2 b = a^3$ implies $ab = a^2$ in any Jordan algebra without any non-zero nilpotent element? It seems to be an open question. The answer is positive in the interesting case of JB–algebras that is in real Jordan algebras equipped with a complete norm satisfying

$$\|ab\| \le \|a\|\,\|b\|, \quad \|a\|^2 \le \|a^2 + b^2\|\,.$$

LEMMA 2. *Let A be a JB–algebra. The algebra A satisfies the condition (P), contains no non-zero nilpotent element and, for $a, b \in A$, $a^2b = a^3$ implies $ab = a^2$.*

P r o o f . It is clear from the definition of a JB–algebra that $\|a^2\| = \|a\|^2$ and therefore a JB–algebra contains no non-zero nilpotent element. The fact that a JB–algebra satisfies condition (P) is the main result of [15] and, in [1], it is proved that, for two elements a, b in a JB–algebra, $a^2b = 0$ implies $ab = 0$, and thus, $ab = a^2$ is a consequence of $a^2b = a^3$.

PROPOSITION 2. *In any JB–algebra, $a \leq b$ if and only if $a^2b = a^3$ is an order relation.*

A partial answer to Q_1 is proved in [9]: If A is a special Jordan algebra whose special universal envelope is an associative algebra without any non-zero nilpotent element, then the J–relation is an ordering on A. The following result is a more practical criterion and it proves, in particular, that the J–relation is an ordering in every Jordan subalgebra of hermitian elements of $\mathcal{L}(\mathrm{H})^J$, H being a Hilbert space.

PROPOSITION 3. *Let A be a $\star$-algebra with proper involution and let A_H be the Jordan algebra of all hermitian elements of A^J. The $\star$-order coincides with the J–relation on A_H and then the J–relation is an ordering on any Jordan subalgebra of A_H.*

P r o o f . In this proof, the $\star$-order is denoted by $\leq^\star$, $\leq$ is the J–relation and $\circ$ is used for the special Jordan product. Let a and b be two hermitian elements of A. If $a \leq^\star b$, then we have $a^2 = ab = ba$. Therefore $a^2 = a \circ b$ and $a^3 = a^2 \circ b$ hold and thus $a \leq b$.

Conversely, assume $a \leq b$. We have $a \circ b = a^2$ and $a^2 \circ b = a^3$. Since the involution is proper, A_H contains no non-zero nilpotent element and, by Lemma 1, $a \circ b^2 = a^3$. Thus:

$$ab + ba = 2a^2, \tag{$\star$}$$

$$a^2b + ba^2 = 2a^3 \quad \text{and} \tag{$\star\star$}$$

$$ab^2 + b^2a = 2a^3. \tag{$\star\star\star$}$$

By using $(\star)$, $2a^3 = a^2b + aba = aba + ba^2$ and thus $a^2b = ba^2$; citing $(\star\star)$, we obtain

$$a^2b = ba^2 = a^3$$

and therefore $aba = a^3$. It follows from $(\star)$ that $ab^2 + bab = 2a^2b$, $bab + b^2a = 2ba^2$ and, as $ba^2 = a^2b$, we have

$$ab^2 = b^2a.$$

Equality $(\star\star\star)$ implies $ab^2 = b^2a = a^3$ and therefore, $(ab - ba)^2 = 0$. Since, $(ab - ba)^2 = -(ab - ba)(ab - ba)^\star$ and the involution is proper, $ab - ba = 0$ and thus, $ab = ba$. It follows from $(\star)$ that $a^2 = ab = ba$ and therefore $a \leq^\star b$.

Now a problem is: Does all the previous order relations have some common properties? All the definitions use solely multiplication but it seems difficult to solve this problem by means of this operation since multiplicative semigroups of $\star$-rings, alternative rings and Jordan algebras are very different. On the other hand, all these algebras are commutative groups for the addition and this motivates the introduction in the next section of orthomodular groups.

2. Orthomodular groups

DEFINITION 1. An orthomodular group is a commutative group G equipped with an order relation $\leq$ such that for any $a, b, c \in G$:

OG_1: $a \leq b \leq b + c \Rightarrow a \leq a + c$,
OG_2: $a \leq b \leq c \Rightarrow c - b \leq c - a$,
OG_3: $a \leq a + b \Rightarrow a \vee b$ exists and $a \vee b = a + b$,
OG_4: $c \leq a + c$ and $c \leq b + c \Rightarrow c \leq a + b + c$.

At the end of this section we will prove that there is abundance of orthomodular groups. Some elementary properties of orthomodular group are collected in the following lemma.

LEMMA 3. *Let G be an orthomodular group.*

1. $a \leq 2a$ *implies* $a = 0$,
2. 0 *is the least element of* G,
3. $a \leq a + b$ *implies* $b \leq a + b$,
4. $a \leq b$ *is equivalent to* $b - a \leq b$,
5. *If* $a \leq b \leq c$, *then* $c - b \leq b - a$.

This lemma shows that the binary relation $a \leq a + b$ is symmetric and this property is the common axiom in all the various definitions of an orthogonality relation. Here, we adopt the following one.

DEFINITION 2. A binary relation $\perp$ on a set E is called an orthogonality relation if

1. $a \perp b$ implies $b \perp a$ (the relation is symmetric),

2. there exists 0 in E such that $0\perp 0$ and, for all $x \in E$, $x\perp x$ implies $x = 0$.

Any orthomodular group carries a canonical orthogonality relation. Precisely:

PROPOSITION 4. *Let G be an orthomodular group.*

1. *The relation $a\perp b$ if and only if $a \leq a + b$ is an orthogonality relation on G.*
2. *$a \leq b$ is equivalent to $a\perp b - a$,*
3. *If $a\perp b$, then $c \leq b$ implies $a\perp c$. Conversely, if $x\perp b$ implies $x\perp a$ and if a and b have a common upper bound, then $a \leq b$.*
4. *$a\perp b$ if and only if there exists $c \in G$ such that $a \leq c$ and $b \leq c - a$.*

P r o o f. Statements 1) and 2) are easy consequences of Lemma 3. If $a\perp b$ and $c \leq b$, then we have $c \leq b \leq a + b$ and, by using OG_1, $c \leq a + c$ and thus $c\perp a$. Now if $x\perp b$ implies $x\perp a$ and if a and b have a common upper bound c, then $c - b\perp b$ and thus $c - b\perp a$. Therefore,

$$a \leq (c - b) + a = (c - b) \vee a \leq c = \big[(c - b) + a\big] + (b - a),$$

and, using OG_1, we have $a \leq a + (b - a) = b$; thus 3) is proved. For 4), the existence of $c (= a + b)$ is clear if $a\perp b$. Conversely, if c exists, then

$$b \leq c - a \leq c = (c - a) + a$$

and therefore, by using OG_4, $b \leq b + a$ which means $a\perp b$.

DEFINITION 3. In an orthomodular group, the order relation $\leq$ and the orthogonality relation $a\perp b$ if and only if a$\leq a + b$ are said to be associated.

Orthogonality relations associated to all the previous orderings will be make explicit in the next section.

Recall the definition of a weak generalized orthomodular poset (abbreviated as WGOMP):

DEFINITION 4. [16] Let $(A, \leq)$ be a poset with the least element 0, such that every interval $[0, a]$ of A is equipped with a unary operation $x \mapsto x^{\perp_a}$. We shall say that A is a weak generalized orthomodular poset if it satisfies the following conditions:

W_1: If $a \in A$, then $\big([0, a], \leq, \perp_a\big)$ is an orthomodular poset.
W_2: If $a \leq b \leq c$, then $a^{\perp_b} = a^{\perp_c} \wedge b$.
W_3: If $a \leq c$, $b \leq c$ and $a \leq b^{\perp_c}$, then the supremum $a \vee b$ exists in A.

We shall write $a \perp b$ if and only if $a \vee b$ exists in A and $a \leq b^{\perp_{a\vee b}}$. If $a \perp b$, then a and b are said to be orthogonal

W_4: If $a, b, c \in A$ with $a \perp b$, $c \perp a$ and $c \perp b$, then $c \perp a \vee b$.

Notice that WGOMPs generalize orthomodular posets in the same way that generalized orthomodular lattices of M. F. Janowitz generalize orthomodular lattices. An interesting property of WGOMPs is the following result [16].

Proposition 5. *Any WGOMP can be embedded as an order ideal in an orthomodular poset. This embedding preserves the orthogonality relation, all existing infima and the supremum of any two orthogonal elements.*

Proposition 6. *Let A be*

- *a $\star$-ring with a proper involution or,*
- *an alternative ring without no nonzero nilpotent element or,*
- *a Jordan algebra satisfying the condition (P) and without nonzero nilpotent element or,*
- *a JB–algebra.*

Equipped with the previous order relations, the additive group of A is an orthomodular group.

Proof. We give a proof for Jordan algebras. Other cases are very similar and only use elementary results about $\star$-rings and alternative rings.

OG_1: Assume $a \leq b \leq b+c$. As $a \leq b+c$, we have $a^3 = a^2b + a^2c$ and $a^2 = ab + ac$. Hence, $a^3 = a^2b$ and $a^2 = ab$ imply $a^2c = ac = 0$ and, therefore, $a^3 = a^2(a+c)$ and $a^2 = a(a+c)$; thus $a \leq a+c$.

OG_2: Assuming $a \leq b \leq c$ we have $bc = b^2$, $b^2c = b^3$, $c^2a = a^3$ and $b^2a = a^3$. It follows that $(c-b)^2(c-a) = c^3 - b^3$ and $(c-b)^3 = c^3 - b^3$; thus $(c-b)^2(c-a) = (c-b)^3$. Similarly, we have $(c-b)(c-a) = (c-b)^2$ and therefore $c - b \leq c - a$.

OG_3: If $a \leq a+b$ then, by lemma 3, $b \leq a+b$ and $a+b$ is an upper bound of $\{a, b\}$. Let c be any upper bound of $\{a, b\}$. As $a^2 = a(a+b)$, $ab = 0$ and we have

$$(a+b)c = ac + bc = a^2 + b^2 = (a+b)^2 .$$

By using $a^2b = ab^2 = 0$, $(a+b)^2c = (a+b)^3$ holds and thus $a+b \leq c$. Finally, $a \vee b$ exists and $a \vee b = a+b$.

OG_4: Assuming $c \leq a+c$ and $c \leq b+c$ we have $c^2a = ca = c^2b = cb = 0$. It follows that $c^2(a+b+c) = c^3$ and $c(a+b+c) = c^2$ and thus $c \leq a+b+c$.

□

The main property of orthomodular groups is given by the following proposition.

PROPOSITION 7. *Let G be an orthomodular group. If every interval $[0,a]$ of G is equipped with the unary operation*

$$x \mapsto x^{\perp_a} = a - x\,,$$

then G is a $WGOMP$.

Proof. W_1: If $x \leq a$ then, by Lemma 3, $x^{\perp_a} = a - x \leq a$ and it is clear that $x^{\perp_a \perp_a} = x$. Now, if $x \leq y \leq a$ the same lemma implies $y^{\perp_a} \leq x^{\perp_a}$. The meet of x and $x^{\perp_a}$ exists by OG_3 since we have $x \leq a = a-(a-x) = x + x^{\perp_a}$ and $x \vee x^{\perp_a} = a$. Hence, any interval $[0,a]$ is an orthocomplemented poset. If $x \leq y^{\perp_a}$, then $x \leq a - y \leq a = (a-y)+y$ and thus, by OG_1, $x \leq x+y$ and $x \vee y$ exists (and $x \vee y = x+y$). For proving the orthomodular law, assume $x \leq y$. Then $x \vee y^{\perp_a}$ exists and $x \vee y^{\perp_a} = x + a - y$. By the de Morgan law, $x^{\perp_a} \wedge y = a - (x+a-y) = y - x$ and thus, $x \vee (x^{\perp_a} \wedge y) = x + y - x = y$.

W_2: Assume that $x \leq a \leq b$. As $x \leq a$, we have $x \vee a^{\perp_b} = x + (b-a)$ and therefore

$$x^{\perp_b} \wedge a = b - \big(x + (b-a)\big) = a - x = x^{\perp_a}$$

and W_2 is proved.

W_3: If $a \leq c$, $b \leq c$ and $a \leq c - b$ then, as $c - b \leq c$ may be written $c - b \leq (c-b)+b$, OG_1 implies $a \leq a+b$ and therefore $a \vee b$ exists.

W_4: Assume $a \perp b$, $c \perp a$ and $c \perp b$. We have

$$c \leq c + a = c \vee a \leq c \vee (a \vee b) = c + (a+b)$$

and so $c \perp a + b = a \vee b$. □

3. Orthogonality relations

In this section, we specify the different orthogonality relations associated to the orderings of Section 1 by the way of the Proposition 4.

PROPOSITION 8. *The orthogonality relations associated to the previous order relations are:*

1. *In a $\star$-ring with a proper involution:*

$$a \perp_1 b \;\Leftrightarrow\; ab^\star = a^\star b = 0\,.$$

2. *In an alternative ring without non-zero nilpotent element:*

$$a \perp_2 b \quad \Leftrightarrow \quad ab = 0\,.$$

3. *In a Jordan algebra satisfying the condition* (P) *and without non-zero nilpotent element:*

$$a \perp_3 b \quad \Leftrightarrow \quad ab = a^2 b = 0\,.$$

4. *In a* JB*–algebra:*

$$a \perp_4 b \quad \Leftrightarrow \quad a^2 b = 0\,.$$

Proof. Use $a \perp b$ if and only if $a \leq a + b$, with each order relation.

Remarks.

1. The relation $\perp_1$ was introduced for $\star$-algebras in [14], page 26, by L. H. Loomis without reference to the $\star$-order. M. Hestenes defined in [12] the same relation for $m \times n$ complex matrices.

2. In [1], the relation $\perp_4$ is defined independently of the J–order.

4. Commutativity

In an orthomodular poset P, two elements a and b commute (or are compatible) if there exist three mutually orthogonal elements of P, a_1, b_1 and c, such that:

$$a = c \vee a_1, \qquad b = c \vee b_1\,.$$

Commutativity of a and b is denoted by $a \leftrightarrow b$. We recall that this notion is the main tool for calculation in orthomodular posets and it plays an important role in the axiomatic of quantum theories [18].

Let G be an orthomodular group, with its canonical structure of $WGOMP$, and let P be the orthomodular poset in which G is embedded as an order ideal by the way of Proposition 5. If two elements a and b of G satisfy $a = c \vee a_1$, $b = c \vee b_1$ where a_1, b_1 and c are mutually orthogonal elements, then a and b commute in P since the embedding preserves the relation of orthogonality and the supremum of orthogonal elements. Therefore, we extend the definition of commutativity to orthomodular groups and, as in orthomodular posets, if a and b commute, then $a \vee b$ and $a \wedge b$ exist and $a \vee b = a + b_1 = b + a_1$, $a \wedge b = c$.

PROPOSITION 9. *1) Let A be an alternative ring without non-zero nilpotent element or a Jordan algebra satisfying condition (P) and without non-zero nilpotent element. Two elements a and b of A commute if and only if*

$$a \wedge b \text{ exists} \quad \text{and} \quad ab = ba = (a \wedge b)^2 .$$

2) Let A be a $\star$-ring with proper involution Two elements a and b of A commute if and only if

$$a \wedge b \text{ exists,} \quad ab^\star = (a \wedge b)(a \wedge b)^\star \quad \text{and} \quad a^\star b = (a \wedge b)^\star (a \wedge b) .$$

Proof. We give a proof for $\star$-rings. Proofs for alternative rings and Jordan algebras use a similar technique.

Assume that $a \wedge b$ exists, $ab^\star = (a \wedge b)(a \wedge b)^\star$, $a^\star b = (a \wedge b)^\star (a \wedge b)$ and let $a_1 = a - a \wedge b$, $b_1 = b - a \wedge b$ and $c = a \wedge b$. We have $c \leq a = a_1 + c$ and $c \leq b = b_1 + c$ and therefore, $c \perp a_1$ and $c \perp b_1$ holds.

$$\begin{aligned} a_1 b_1^\star &= (a - a \wedge b)(b - a \wedge b)^\star = \big(a - (a \wedge b)\big)\big(b^\star - (a \wedge b)^\star\big) = \\ &= ab^\star - a(a \wedge b)^\star - (a \wedge b)b^\star + (a \wedge b)(a \wedge b)^\star = \\ &= ab^\star - (a \wedge b)(a \wedge b)^\star - (a \wedge b)(a \wedge b)^\star + (a \wedge b)(a \wedge b)^\star = \\ &= 0 \end{aligned}$$

Similarly, we have $a_1^\star b_1 = 0$. Thus a_1 and b_1 are orthogonal elements and a and b commute. Conversely, if a and b commute, then there exist mutually orthogonal elements a_1, b_1 and c such that $a = c \vee a_1 = c + a_1$ and $b = c \vee b_1 = c + b_1$. We know that $c = a \wedge b$ and $cc^\star = (a - a_1)(b - b_1)^\star = ab^\star$ since $ab_1^\star = a_1 b^\star = a_1 b_1^\star = 0$. In the same way, we have $c^\star c = a^\star b$.

Remarks.

1) Commutativity of elements a and b may be also characterized by means of their join as follows:

$$a \leftrightarrow b \quad \Leftrightarrow \quad a \vee b \text{ exists} \quad \text{and} \quad (a \vee b)^2 = a^2 + b^2 - ab$$

in alternative rings and Jordan algebras,

$$\begin{aligned} a \leftrightarrow b \quad \Leftrightarrow \quad a \vee b \text{ exists,} \quad & (a \vee b)(a \vee b)^\star = aa^\star + bb^\star - ab^\star \\ \text{and} \quad & (a \vee b)^\star (a \vee b) = a^\star a + b^\star b - a^\star b \end{aligned}$$

in $\star$-algebras.

2) If a and b commute then:

$$(a \vee b)^2 + (a \wedge b)^2 = a^2 + b^2$$

in case of an alternative ring or a Jordan algebra and

$$(a \vee b)(a \vee b)^{\star} + (a \wedge b)(a \wedge b)^{\star} = aa^{\star} + bb^{\star}$$

for $\star$-algebras.

3) In a Jordan ring A, the multiplication operator $T_a : A \mapsto A$ is defined, for each $a \in A$, by $T_a(x) = ax$. Two elements a and b of A *operator commute* if $T_a \circ T_b = T_b \circ T_a$ or, equivalently, if $a(xb) = (ax)b$ for all x in A. If a and b commute, then it is easy to check that $a^2 b = a(ab)$ and this equality is equivalent, in a JB–algebra, to a and b as operators commute [4]. Thus, for JB–algebras, commutativity implies operator commutativity. We will prove, in the next section, that the two concepts are equivalent for idempotents in unital Jordan algebras.

5. Idempotents

In this section, we assume that all Jordan rings satisfy $2a = 0$ implies $a = 0$. A lemma will be useful for the proof of the first proposition.

Lemma 4. *Let A be a Jordan ring. If p and q are two idempotents such that $p = pq$, then, for all y in A, $(py)q = p(yq)$; that is p and q operator commute.*

P r o o f. If, in the identity $(x, y, x^2) = x(yx^2) - (xy)x^2 = 0$, $x + z$ replaces x, then we obtain

$$(x, y, z^2) + (z, y, x^2) + 2(x, y, xz) + 2(z, y, xz) = 0 .$$

By choosing $x = p$ and $z = q$ we have $2(q, y, p) = 0$ and thus $(py)q = p(yq)$.

Proposition 10. *1) Let A be an alternative ring or a Jordan ring. The binary relation*

$$p \leq q \Leftrightarrow p = pq = qp$$

is an order relation on the set of all idempotents of A which is a $WGOMP$ and an orthomodular poset in the unital case.

2) Let A be a $\star$-ring. The binary relation

$$p \leq q \Leftrightarrow p = pq$$

is an order relation in the set of all projections of A which is a $WGOMP$ and an orthomodular poset in the unital case.

P r o o f. 1) Clearly, the relation is reflexive and antisymmetric. For proving transitivity of this relation recall two Moufang identities which are valid in any alternative ring:

$$(axa)y = a\big[x(ay)\big]\,, \tag{M}$$
$$y(axa) = \big[(ya)x\big]a\,. \tag{M*}$$

Now, let p, q and r be three idempotents of an alternative ring such that $p \le q$ and $q \le r$. We have

$$pr = (qp\,q)r = q\big[p(qr)\big] = q(p\,q) = p$$

by using (M). Similarly, (M^*) implies $rp = p$ and thus $p \le r$ is proved.

Now if p, q and r are three idempotents of a Jordan ring such that $p \le q$ and $q \le r$ then, by using Lemma 4, $(p\,r)q = p(rq) = p\,q = p$ and $p\,r = (qp)r = q(p\,r)$; thus $p = p\,r$ and $p \le r$ is proved.

The proof that the set of all idempotents forms a $WGOMP$ is routine.

2) This result is well-known (see [3], Exercise 11, page 54).

Remarks. 1) The previous proposition shows that the binary relation $p \le q \Leftrightarrow p = p\,q = qp$ is an order relation in the set of all idempotents of a ring and so it generalizes a result of [8] or [13].

2) In [9], it is proved that $p \le q \Leftrightarrow p = p\,q = qp$ is an order relation in the set of idempotents of a Jordan algebra.

3) Notice that in a ring with non-zero nilpotent element the binary relation $p \le q \Leftrightarrow p = qp$ is not always an antisymmetric relation. As an example consider the two idempotents p and q in the ring of 2×2 matrices with real coefficients:

$$p = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that $q = qp$ and $p = p\,q$. Recall that in an alternative ring and, in particular in a ring, without non-zero nilpotent element every idempotent is central and thus, the binary relation $p \le q \Leftrightarrow p = qp$ is an order relation. In this case, the set of all idempotents is a Boolean algebra.

Proposition 11. *1) Let A be an alternative ring without non-zero nilpotent element or a Jordan algebra without any non-zero nilpotent element and satisfying the condition (P). If p is an idempotent of A and if $a \le p$, then a is an idempotent.*

2) Let p be a projection in a $\star$-ring with a proper involution. If $a \le p$, then a is a projection

The proof is easy. □

Remark. By adding some hypotheses, every interval $[0, p]$, where p is an idempotent or a projection, becomes an orthomodular lattice which contains an isomorphic copy of any interval $[0, a]$. More precisely, the following similar results can be proved:

1) Let A be a Rickart $\star$-ring ordered by the $\star$-order. The set of all projections of A constitutes an orthomodular lattice and, for any a in A, the interval $[0, a]$ is an orthomodular lattice isomorphic to a subalgebra of $[0, a'']$, where a'' is the range projection of a [10].

2) Let A be a JBW–algebra (that is a JB–algebra which is a dual Banach space) ordered by the J–order. The set of all idempotents of A is a complete orthomodular lattice [20] and for any a in A, the interval $[0, a]$ is an orthomodular lattice isomorphic to a subalgebra of $[0, a'']$, where a'' is the range idempotent of a [6].

Recall that in a JB–algebra an important ordering is that defined by the cone of squares [11]:

$$a \preceq b \quad \text{if and only} \quad b - a = x^2 \quad \text{for some} \quad x \in A.$$

The following result seems to be know but, since we were was unable to find any reference, we shall include a proof.

PROPOSITION 12. *Let p and q be two idempotents of a JB–algebra. Then $p \leq q$ (where $\leq$ denotes the J–order) is equivalent to $p \preceq q$.*

Proof. If $p = pq$, then $(q-p)^2 = q+p-2pq = q-p$; thus $q-p$ is a square and $p \preceq q$ holds. Conversely, assume $p \preceq q$. We have $0 \preceq q-p = (q-1)-p+1$, which implies $p \preceq (1-q)+p \preceq 1$. Since the mapping U_p is positive for $\preceq$ ([4], Prop. 3.3.6), $U_p(p) \preceq U_p(1-q)+U_p(p) \preceq U_p(1)$; that is $p \preceq U_p(1-q)+p \preceq p$ and therefore $U_p(1-q) = 0$. By using identity 3 we obtain $U_{1-q}(p) = 0$ and choosing $x = p$ and $y = 1-q$ in (2) yields to $p(1-q) = 0$; thus $p = pq$ and $p \leq q$ is proved.

PROPOSITION 13. *Let A be a unital Jordan algebra without non-zero nilpotent element and satisfying the condition (P). Two idempotents of A commute if and only if their operators commute.*

Proof. If p and q commute, then p and q as operators commute. Now, assume that p and q operator commute. By [4], Lemma 2.5.5, p and q generate an associative subalgebra of A and this implies that pq is an idempotent, $p(pq) = pq$ and $q(pq) = pq$. Thus pq is a lower bound of $\{p, q\}$. Let a be an other lower bound. By Proposition 11, a is an idempotent and, since it commutes with q, its operator also commutes with q. Therefore, $a(pq) = (ap)q = aq = a$ and $a \leq pq$ holds. Hence, $p \wedge q$ exists and $p \wedge q = pq$. Now, Proposition 9 proves that p and q commute.

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