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ALEXANDROV AND KOLMOGOROV CONSISTENCY THEOREM FOR MEASURES WITH VALUES IN PARTIALLY ORDERED GROUPS

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ABSTRACT. Alexandrov and Kolmogorov theorems for Riesz space valued measures were discussed in [2] and [6]. The relations between the regularity and countable additivity for measures with values in partially ordered vector spaces were studied in [12]. The goal of this note is to suggest the concept of the regularity of μ which gives the desired results for partially ordered group valued measures.

1. Introduction

Let $(G,+,\leqq)$ be a commutative partially ordered group. G is said to be *monotone σ-complete* if, whenever (a_n) $(n = 1,2,...)$ is an upper bounded, monotone increasing in G, then it has a least upper bound $\bigvee a_n \in G$. If, for each upper bounded, upward directed family (a_λ) in G, there exists a least upper bound $\bigvee a_\lambda \in G$, then G is said to be *monotone complete.* The set of nonnegative elements in G will be denoted by G^+. Denote by $\mathbb{N}$ the set of all positive integers.

Let $\mathcal{R}$ be a ring of subsets of a nonempty set X and μ be a positive, finitely additive G-valued mapping on $\mathcal{R}$, i.e., $\mu : \mathcal{R} \to G^+$, i.e., $\mu(A) \geq 0$, for any $A \in \mathcal{R}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for all $A, B \in \mathcal{R}$ if $A \cap B = \emptyset$. (Due to additivity $\mu(\emptyset) = 0$.) Recently, (see [4]) the term "charge" is used for such mapping μ, of course, "positive" charge in this case.

If μ is a G^+-valued additive mapping on a ring $\mathcal{R}$, there are several ways how to define its countable additivity. We use the following definition: A mapping $\mu : \mathcal{R} \to G^+$ is *countably additive* if

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigvee\Big\{\sum_{i=1}^{n} \mu(A_i) \mid n = 1,2,\dots\Big\},$$

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whenever (A_i) is a sequence of disjoint sets in $\mathcal{R}$. Instead of $\bigvee\Big\{\sum\limits_{i=1}^{n} a_i \mid n = 1,2,\dots\Big\}$ we will write $\sum\limits_{i=1}^{\infty} a_i$.

Referring to F l o y d 's paper [1], J. D. M. W r i g h t pointed out that the theory based on this concept of countable additivity differs from the theory of topological group valued measures. The basic fact is that there exist partially ordered groups which do not admit any Hausdorff group topology σ-compatible with the order. (A group topology $\mathcal{T}$ is σ-compatible with the order if the following implication holds:

$$\text{if} \quad (a_n) \nearrow a \ \text{(order convergence)},$$

$$\text{then } (a_n) \xrightarrow{\mathcal{T}} a \ \text{(convergence in the topology } \mathcal{T}\text{)}.$$

It is known that for basic results for Riesz space-valued measures the following property of σ-distributivity of a range space plays the central role. As we are going to discuss group valued measures the weak σ-distributivity is presented as the condition of a lattice group. A σ-complete lattice group G is weakly σ-distributive if

$$\bigwedge\Big\{\bigvee_{i=1}^{\infty} a_{i\,\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N}\Big\} = 0\,,$$

whenever $(a_{i\,j})$ is a bounded from above double sequence in G such that for each $i \in \mathbb{N}$, $a_{i\,j} \searrow 0 \ (j \to \infty)$.

For partially ordered groups without the lattice structure the following concept of regularity was used in [7]. A monotone σ-complete partially ordered group G is a regular group if

$$\bigwedge\Big\{\sum_{i=1}^{\infty} a_{i\,\varphi(i)} \mid \varphi\colon \mathbb{N} \to \mathbb{N}\Big\} = 0\,,$$

whenever $(a_{i\,j})$ is a double sequence in G such that $a_{i\,j} \searrow 0 \ (j \to \infty)$ for each $i \in \mathbb{N}$. The relations between these concepts were studied in [8].

Let us return to G^+–valued additive mappings. The following lemma states the simple relation between continuity and countable additivity of μ.

LEMMA 1. *If $\mu : \mathcal{R} \to G^+$ is finitely additive, then the following conditions are equivalent:*

(1) *μ is countably additive on $\mathcal{R}$.*
(2) *μ is continuous from above at $\emptyset$.*
(3) *μ is continuous from below at every $A \in \mathcal{R}$.*

The proof is elementary; if (a_n) is a decreasing sequence of nonnegative elements in G, then $\bigvee(a - a_n) = a - \bigwedge a_n$. □

2. Alexandrov theorem

A classical theorem of A. D. Alexandrov states that when μ is nonnegative, finitely additive, and regular on the ring $\mathcal{R}$ of subsets of a compact Hausdorff space X, then μ is countably additive on $\mathcal{R}$. The generalization of Alexandrov theorem for vector-valued measures was done in [6] and [2] for Riesz space-valued measure, i.e., in the case when the range of μ is a vector lattice. The concept of regularity of vector valued measures was studied earlier in papers [10, 11, 12] of J. D. M. Wright who proved the extension theorem under the assumption of regularity of μ. All the formulations of the regularity in papers we have just mentioned are based on the lattice structure of the range space. For example, in [10] the inner regularity means that

$$\mu(E) = \bigvee \{ \mu(F) \colon \exists C \in \mathcal{C},\ F \subset C \subset E \},$$

where $\mathcal{C}$ is a system of closed (i.e., compact) subsets in the compact Hausdorff space X. Authors in [6] and [2] used a more abstract and delicate formulation of (inner) regularity of μ. First, they used the concept of (abstract) compact system $\mathcal{K}$ (a nonempty system $\mathcal{K} \subset 2^X$ is a compact one if $\bigcap\limits_{i=1}^{\infty} K_i = \emptyset$ $(K_i \in \mathcal{K},\ i = 1, 2, \dots)$ implies that there exists $n_0 \in \mathbb{N}$ such that $\bigcap\limits_{i=1}^{n_0} K_i = \emptyset$), which plays the role of a system of compact sets in X.

Secondly, they used behavior of doubled decreasing sequences tending to zero to express the fact that $\mathcal{K}$ approximates $\mathcal{R}$. We point out that both authors discussed Riesz space-valued measures but what they really utilized was the lattice structure of the range of μ. That is the reason why it is possible to formulate and to prove all their results also for lattice group valued measures. In the next we will use the concept of regular group so to avoid misunderstanding, following [3], instead of (inner) regularity of μ (with respect to a compact system) we will say that μ is compact.

Definition 1. Let G be a commutative lattice group. Let $\mathcal{R}$ be a ring of subsets of X and let $\mathcal{K}$ be a compact system in X. A mapping $\mu \colon \mathcal{R} \to G$ is said to be $\bigvee$-*compact*, if for any $E \in \mathcal{R}$, there exists a bounded double sequence (a_{ij}) in G, $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$, such that for any $\varphi \colon \mathbb{N} \to \mathbb{N}$ there exist $C \in \mathcal{K}$ and $F \in \mathcal{R}$ such that $F \subset C \subset E$ and

$$\mu(E \setminus F) \leqq \bigvee_{i=1}^{\infty} a_{i\,\varphi(i)}.$$

The sense of this definition arises in the case of a weakly σ-distributive lattice group when the infimum of elements $\bigvee_{i=1}^{\infty} a_{i\,\varphi(i)}$ through all φ, $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a zero.

Our goal is to abandon the lattice structure of the range of μ so we offer the following modification of the compactness of μ.

DEFINITION 2. Let G be a commutative partially ordered group. Let $\mathcal{R}$ be a ring of subsets of X and let $\mathcal{K}$ be a compact system in X. A mapping $\mu\colon \mathcal{R} \to G$ is said to be *Σ-compact* if for any $E \in \mathcal{R}$ there exists a double sequence (a_{ij}) in G, $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$, such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exist $C \in \mathcal{K}$ and $F \in \mathcal{R}$ such that $F \subset C \subset E$ and

$$\mu(E \setminus F) \leqq \sum_{i=1}^{\infty} a_{i\,\varphi(i)}\,.$$

The comparison between these concepts in the case of lattice-ordered groups is given in the following proposition.

PROPOSITION 1. *Let G be a σ-complete lattice group. If $\mu\colon \mathcal{R} \to G^+$ is additive, then μ is $\vee$-compact if and only if it is Σ-compact.*

Proof. It is evident that $\vee$-compactness implies Σ-compactness. For the reverse implication we use the lemma from [9]:

If (a_{ij}) is a double sequence in G such that $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$, then to every positive $b \in G$ there exists a bounded double sequence (b_{ij}) in G, $b_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$ and such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ it holds

$$b \wedge \left(\sum_{i=1}^{\infty} a_{i\,\varphi(i)} \right) \leqq \bigvee_{i=1}^{\infty} b_{i\,\varphi(i)}\,. \qquad \square$$

The concept of inner regularity used in [5] is more restrictive than those from Definition 1 and Definition 2. It is easy to see that if μ is inner regular in the sense of [5], then μ is $\vee$-compact. On the other hand, let us consider (inner) regularity of μ in the sense of [10]. Suppose that G is a complete lattice and μ is $\vee$-compact (with respect to a compact system $\mathcal{C}$). If G is weakly σ-distributive, then

$$\mu(E) = \bigvee \{\, \mu(F)\colon \exists C \in \mathcal{C},\ F \subset C \subset E \}\,.$$

To prove it, let us set $a = \mu(E) - \bigvee\{\mu(F)\colon\ \exists C \in \mathcal{C},\ F \subset C \subset E\}$. It is obvious that a is nonnegative and

$$\begin{aligned} a &= \mu(E) - \bigvee \{\mu(F)\colon \exists C \in \mathcal{C},\ F \subset C \subset E\} = \\ &= \bigwedge \{\mu(E \setminus F)\colon \exists C \in \mathcal{C},\ F \subset C \subset E\}\,. \end{aligned}$$

Due to the assumption, there exists a bounded double sequence (a_{ij}) in G, $a_{ij} \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$, such that for any $\varphi_0 \colon \mathbb{N} \to \mathbb{N}$ there exist $C_0 \in \mathcal{C}$ and $F_0 \in \mathcal{R}$ such that $F_0 \subset C_0 \subset E$ and

$$\mu(E \setminus F_0) \leqq \bigvee_{i=1}^{\infty} a_{i\,\varphi_0(i)} .$$

We get

$$a = \bigwedge \{ \mu(E \setminus F) \colon \exists C \in \mathcal{C},\ F \subset C \subset E \} \leqq \bigvee_{i=1}^{\infty} a_{i\,\varphi_0(i)}$$

for any φ_0, $\varphi_0 \colon \mathbb{N} \to \mathbb{N}$, i.e., $a \leq \bigwedge \Big\{ \bigvee\limits_{i=1}^{\infty} a_{i\,\varphi(i)} \mid \varphi \colon \mathbb{N} \to \mathbb{N} \Big\} = 0$ due to the weak σ-distributivity of G. Consequently,

$$\mu(E) = \bigvee \{ \mu(F) \colon \exists C \in \mathcal{C},\ F \subset C \subset E \} .$$

For the proof of Alexandrov theorem we will need the following computational lemma.

LEMMA 2. *Let (a^n_{ij}) be a triple sequence in G such that $a^n_{ij} \searrow 0$ $(j \to \infty)$ for each $n, i \in \mathbb{N}$. Then there exists a double sequence (b_{ij}) in G, $b_{ij} \searrow 0$ $(j \to \infty)$, such that for any $\varphi \colon \mathbb{N} \to \mathbb{N}$*

$$\sum_{j=1}^{n} \sum_{i=1}^{\infty} a^j_{i\,\varphi(j+i-1)} \leqq \sum_{k=1}^{\infty} b_{k\,\varphi(k)} .$$

Proof. Set $b_{kj} = \sum\limits_{r=1}^{k} a^r_{k-r+1\,j}$. The relations

$$a^1_{k-1+1\,j} \searrow 0 \qquad (j \to \infty),$$
$$a^2_{k-2+1\,j} \searrow 0 \qquad (j \to \infty),$$

imply

$$(a^1_{k-1+1\,j} + a^2_{k-2+1\,j}) \searrow 0 \qquad (j \to \infty),$$

consequently, we get $\sum\limits_{r=1}^{k} a^r_{k-r+1\,j} \searrow 0$ $(j \to \infty)$, so that $b_{kj} \searrow 0$ $(j \to \infty)$. It is easy to verify that for each $m \in \mathbb{N}$ we get

$$\sum_{j=1}^{m} \sum_{i=1}^{m} a^j_{i\,\varphi(j+i-1)} \leqq \sum_{k=1}^{2m-1} b_{k\,\varphi(k)} ,$$

so the desired inequality holds. □

Theorem 1 (Alexandrov). *Let $\mu : \mathcal{R} \to G^+$ be additive and Σ-compact (with respect to a compact system $\mathcal{K}$) on the ring $\mathcal{R}$. If G is a monotone σ-complete, regular partially ordered group, then μ is countably additive.*

Proof. With respect to Lemma 1 it is sufficient to show that μ is continuous from above at $\emptyset$. Let (E_n) be decreasing to $\emptyset$, $E_n \in \mathcal{R}$, for $n \in \mathbb{N}$. Since μ is Σ-compact with respect to $\mathcal{K}$, for each E_n, there exists a triple (a^n_{ij}), $a^n_{ij} \searrow 0$, $(j \to \infty)$ such that for any φ, $\varphi : \mathbb{N} \to \mathbb{N}$, there exist $K_n \in \mathcal{K}$ and $F_n \in \mathcal{R}$ such that $F_n \subset K_n \subset E_n$ and

$$\mu(E_n \setminus F_n) \leqq \sum_{i=1}^{\infty} a^n_{i\,\varphi(i)} .$$

Short inspection shows that from compactness of μ we can conclude the following: there exists a triple (a^n_{ij}) in G, $a^n_{ij} \searrow 0$ $(j \to \infty)$, such that for any φ, $\varphi : \mathbb{N} \to \mathbb{N}$, there exist $K_n \in \mathcal{K}$ and $F_n \in \mathcal{R}$ such that $F_n \subset K_n \subset E_n$ and

$$\mu(E_n \setminus F_n) \leqq \sum_{i=1}^{\infty} a^n_{i\,\varphi(n+i-1)} .$$

Since $\bigcap\limits_{i=1}^{\infty} E_i = \emptyset$, we get $\bigcap\limits_{i=1}^{\infty} K_i = \emptyset$ and due to compactness of $\mathcal{K}$ there exists $n_0 \in \mathbb{N}$, such that $\bigcap\limits_{i=1}^{n_0} K_i = \emptyset$. From this we have $\bigcap\limits_{j=1}^{n} F_j = \emptyset$, if $n \geqq n_0$. Hence for $n \geqq n_0$ we get

$$\mu(E_n) = \mu\Bigl(E_n \setminus \bigcap_{j=1}^{n} F_j\Bigr) \leqq \mu\Bigl(\bigcup_{j=1}^{n} (E_j \setminus F_j)\Bigr) \leqq \sum_{j=1}^{n} \sum_{i=1}^{\infty} a^j_{i\,\varphi(j+i-1)} ,$$

according to Lemma 2 there exists (b_{ij}), $b_{ij} \searrow 0$ $(j \to \infty)$ such that

$$\mu(E_n) \leqq \sum_{j=1}^{n} \sum_{i=1}^{\infty} a^j_{i\,\varphi(j+i-1)} \leqq \sum_{i=1}^{\infty} b_{i\,\varphi(i)} .$$

Sets K_n, F_n depend on φ so the index n_0 also depends upon φ. Anyway, for $\bigwedge\limits_{n=1}^{\infty} \mu(E_n)$ we can write

$$\bigwedge_{n=1}^{\infty} \mu(E_n) \leqq \bigwedge \Bigl\{ \sum_{i=1}^{\infty} b_{i\,\varphi(i)} \mid \varphi \colon \mathbb{N} \to \mathbb{N} \Bigr\}$$

and from the regularity of G we get $\bigwedge\limits_{n=1}^{\infty} \mu(E_n) = 0$, so that μ is continuous at $\emptyset$, i.e., μ is countably additive. $\square$

Application: Kolmogorov's consistency theorem

Let I be an arbitrary nonempty set and denote by $\mathcal{F}$ the set of all finite subsets of I. Let $(X_i)_{i\in I}$ be a system of topological spaces and let $X^f = \prod_{i\in f} X_i$ $(f \in \mathcal{F})$ be the product space (i.e., X^f is endowed with the product topology). Suppose that for each $f \in \mathcal{F}$, $\mathcal{K}^f$ is the system of all compact subsets in X^f and $\mu^f : \mathcal{A}^f \to G$ is a compact measure on an algebra $\mathcal{A}^f$, $\mathcal{K}^f \subset \mathcal{A}^f$ and μ^f is compact with respect to $\mathcal{K}^f$.

Finally, denote by Π_f the projection from $\prod_{i\in I} X_i$ into X^f and by Π_{gf}, where $f, g \in \mathcal{F}$, $f \subset g$, the projection from X^g into X^f. Let $\mathcal{A}$ be the algebra of subsets of $\prod_{i\in I} X_i$ generated by the system $\{\Pi_f^{-1}(E) \colon E \in \mathcal{A}^f,\ f \in \mathcal{F}\}$.

Theorem 2 (Kolmogorov). *Under the notation given above suppose that systems*

$$\{\Pi_f^{-1}(\mathcal{A}^f) \colon f \in \mathcal{F}\}, \quad \Pi_f^{-1}(\mathcal{A}^f) = \{\Pi_f^{-1}(E) \colon E \in \mathcal{A}^f\}$$

are directed upwards by set inclusion, i.e., for any $f, g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $\Pi_f^{-1}(\mathcal{A}^f) \subset \Pi_h^{-1}(\mathcal{A}^h)$ and $\Pi_g^{-1}(\mathcal{A}^g) \subset \Pi_h^{-1}(\mathcal{A}^h)$. Let G be a monotone σ-complete, regular partially ordered group and $\{\mu^f : f \in \mathcal{F}\}$ be a consistent system of compact measures, i.e., if $f, g \in \mathcal{F}$ and $f \subset g$, then $\mu^f(E) = \mu^g\big(\pi_{g\,f}^{-1}(E)\big)$ for any $E \in \mathcal{A}^f$. Then there exists a measure μ, $\mu : \mathcal{A} \to G$ such that

$$\mu\big(\Pi_f^{-1}(E)\big) = \mu^f(E)$$

for any $f \in \mathcal{F}$, $E \in \mathcal{A}^f$. Moreover, if G is an o-separable group, then there exists a measure μ^* with the above property on the σ-algebra $\mathcal{S}(\mathcal{A})$ generated by $\mathcal{A}$.

Proof. In [2] it is shown that μ defined on $\mathcal{A}$ by $\mu(A) = \mu^f\big(\Pi_f(A)\big)$ is unambiguously defined ([2], Lemma 4). The fact that $\mathcal{K} = \big\{\Pi_f^{-1}(K) \colon K \in \mathcal{K}^f,\ f \in \mathcal{F}\big\}$ is a compact system is proved in [5]. The rest of the proof is completely analogous with the proof in [2]. Let us recall that G is o-separable if every nonempty subset H possessing a supremum contains an at most countable subset possessing the same supremum as H (realize that in this case G is monotone complete). The last part of the Theorem is the consequence of Theorem 3, p. 375 in [7]. □

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