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REMARKS ON BUCK'S MEASURE DENSITY

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ABSTRACT. In this paper, Buck's measure density is investigated. Some properties of this set function are proved.

Introduction

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Z}$ the set of all integers. In [1], the measure density of a set $S \subset \mathbb{N}$ has been introduced in the following way: For a nonnegative integer and $m \in \mathbb{N}$ we put

$$a + \langle m \rangle = \{a + m \cdot n;\ n = 0, 1, 2, \dots\}.$$

If $S \subset \mathbb{N}$, then the value

$$\mu^*(S) = \inf\left\{\sum_{i=1}^{k} \frac{1}{m_i};\ S \subset \bigcup_{i=1}^{k} a_i + \langle m_i \rangle,\ m_i \in \mathbb{N}\right\}$$

will be called the *measure density* of the set S.

The purpose of this paper is to describe some properties of the function μ^*.

In 1962, E. V. Novoselov [5] has constructed a metric ring of polyadic numbers, as a compactification of the ring of integers $\mathbb{Z}$, with a special metric. Let us denote this ring by Ω. If $m \in \mathbb{N}$, then the symbol (m) will denote the principal ideal in Ω, generated by m. Similarly for $a \in \mathbb{Z}$ we put

$$a + (m) = \{a + \gamma \cdot m;\ \gamma \in \Omega\}.$$

In [5], the Haar probability Borel measure P on Ω is also investigated. In this paper, it is proved that

$$P\big(a + (m)\big) = \frac{1}{m}, \quad m \in \mathbb{N}. \tag{1}$$

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From the investigation in [5] we also have that the set $a+(m)$, $m \in \mathbb{N}$, $a \in \mathbb{Z}$, is closed and open (see [5]).

In [5], it is proved:

THEOREM A. *Let $\alpha \in \Omega$, and $m \in \mathbb{N}$. Then there exist unique elements $\beta \in \Omega$ and $j \in Z$, such that $0 \leq j < m$ and*

$$\alpha = m \cdot \beta + j .$$

Theorem A shows that the relation of divisibility can be naturally extended to Ω. For the extension of the metric d on Ω then one has

$$d(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\alpha - \beta)}{2^n} \tag{2}$$

for $\alpha, \beta \in \Omega$, where

$$\varphi_n(\gamma) = \begin{cases} 1, & \text{if} \quad n \nmid \gamma , \\ 0, & \text{if} \quad n \mid \gamma . \end{cases}$$

Thus the convergence in Ω can be characterized as follows: Let $\{\alpha_n\}$ be a sequence of elements of Ω and $\alpha \in \Omega$. Then $\alpha_n \to \alpha$ if and only if for every positive integer M there exists an index n_0 such that for $n \geq n_0$ we have

$$\alpha_n \equiv \alpha \pmod{M!} . \tag{3}$$

We shall also need the well-known Riemann zeta function ζ defined as follows: For $\alpha > 1$ we put

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} .$$

From [2, page 246] we have: If $\alpha > 1$, then

$$\frac{1}{\zeta(\alpha)} = \prod_p \Bigl(1 - \frac{1}{p^\alpha}\Bigr) . \tag{4}$$

If $S \in \mathbb{N}$, then the value of the limit (if it exists)

$$d(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{\substack{k \leq n \\ k \in S}} 1$$

will be called the asymptotic density of the set S. Clearly

$$d(S) \le \mu^*(S) \tag{5}$$

if $d(S)$ exists.

In the paper [7], the following formula for evaluation of the measure density is proved. For $a, b \in \mathbb{N}$, denote by $a \mod b$ the least nonnegative remainder of a after division by b. For the set $S \in \mathbb{N}$ and $b \in \mathbb{N}$ we put

$$S \pmod b = \{s \pmod b\,;\ s \in S\}$$

and

$$R(S, b) = \#S \pmod b\,.$$

THEOREM B. *Let $\{B_n\}$ be a sequence of positive integers for which the following condition is satisfied:*

(i) *For every $d \in \mathbb{N}$ there exists n_0 such that for $n > n_0$ we have $d \,|\, B_n$.*

Then

$$\mu^*(S) = \lim_{n \to \infty} \frac{R(S, B_n)}{B_n}$$

for every $S \subset \mathbb{N}$.

Proof. See [7, p. 17]. □

1. Buck's measure density and the measure on Ω.

In this part, we establish one connection between the measure P on Ω and the measure density.

Let us denote for $S \subset \mathbb{N}$ by $\overline{S}$ the topological closure of S in the space Ω.

THEOREM 1. *For $S \subset \mathbb{N}$ we have*

$$\mu^*(S) = P(\overline{S})\,.$$

Proof. Let $\{B_n\}$ be a sequence for which (i) is satisfied and moreover $B_n | B_{n+1}$. (For instance $B_n = n!$, $n = 1, 2, \dots$.)

Put

$$S + (B_n) = \bigcup_{s \in S} s + (B_n), \quad n = 1, 2, \dots\,.$$

Then $S+(B_n) \supset S+(B_{n+1})$, $n=1,2,\dots$. From (3) and Theorem A it follows

$$\overline{S} = \bigcap_{n=1}^{\infty} S+(B_n).$$

Thus the upper semi-continuity of measure gives us

$$P(\overline{S}) = \lim_{n\to\infty} P\big(S+(B_n)\big). \tag{6}$$

Clearly the set $S+(B_n)$ has a disjoint decomposition

$$S = \bigcup_{a\in S \pmod{B)_n}} a+(B_n).$$

Thus (1) implies that

$$P\big(S+(B_n)\big) = \frac{R(S,B_n)}{B_n}.$$

Therefore (6) and Theorem B implies the assertion of Theorem 1. The proof is complete. □

Denote by D the set of all primes and by Q_k for $k=2,3,\dots$ the set of all $a\in\mathbb{N}$ such that there is no prime number $p\in D$ with $p^k|a$. Clearly Q_2 will denote the well known set of square-free integers. We shall now use Theorem 1 for the evaluation of $\mu^*(Q_k)$.

In [6, p. 65] it is proved that

$$d(Q_k) = \frac{1}{\zeta(k)} \tag{7}$$

for $k=2,3,\dots$.

THEOREM 2. *For $k=2,3,\dots$ we have*

$$\mu^*(Q_k) = \frac{1}{\zeta(k)}.$$

P r o o f. Clearly for $k=2,3,\dots$

$$Q_k \subset \Omega \setminus \bigcup_{p\in D} (p^k).$$

The set $\bigcup\limits_{p\in D}(p^k)$ is open thus its complement is closed. Therefore

$$\overline{Q}_k \subset \Omega \setminus \bigcup_{p\in D}(p^k).$$

Now, Theorem 1 implies that

$$\mu^*(Q_k) = P(\overline{Q}_k) \leq P\Big(\Omega \setminus \bigcup_{p\in D}(p^k)\Big). \tag{8}$$

But from the exclusion-inclusion principle we have

$$P\Big(\Omega \setminus \bigcup_{p\in D}(p^k)\Big) = \prod_{p\in D}\Big(1 - \frac{1}{p^k}\Big) = \frac{1}{\zeta(k)}.$$

Thus (5), (7) and (8) implies Theorem 2. The proof is complete. □

Let us conclude this part with one remark. The set $S \in \mathbb{N}$ is called measurable if

$$\mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1.$$

In [7] it is proved that every measurable set S has asymptotic density and $d(S) = \mu^*(S)$. It is a natural question whether the equation $d(S) = \mu^*(S)$ implies the fact that S is a measurable set. Theorem 2 gives us the negative answer to this question, although $d(Q_k) = \mu^*(Q_k)$. In [8] is proved that this set does not contain any infinite arithmetic progression, thus Theorem B implies $\mu^*(\mathbb{N} \setminus Q_k) = 1$ for $k = 2, 3, \dots$ and so Q_k is not measurable.

2. Remarks on a certain arithmetic function.

Theorem 1 implies the following property of sets having the measure density 1.

Proposition 1. *If a set $S \subset \Omega$ has the measure density 1, then it is dense in Ω.*

We shall say that an arithmetic function f is *polyadically* continuous if, for every $\varepsilon > 0$, there exists a positive integer N_0 such that for each integer a, b it holds

$$a \equiv b \pmod{N_0!} \Longrightarrow \big|f(a) - f(b)\big| < \varepsilon.$$

Now, let us define an arithmetic function h_α for a real α in the following way:

$$h_\alpha(n) = \prod_{p/n}\Big(1 - \frac{1}{p^\alpha}\Big), \quad n = 1, 2, \dots.$$

LEMMA 1. *For $\alpha > 1$, the arithmetic function h_α is polyadically continuous.*

Proof. Let $a \equiv b \pmod{k!}$. Then in the representation of the numbers a, b as a product of prime powers there appear the same primes which are not greater than k.

Thus, we can put

$$K = \prod_{\substack{p/a \\ p \le k}} \Big(1 - \frac{1}{p^\alpha}\Big) = \prod_{\substack{p/b \\ p \le k}} \Big(1 - \frac{1}{p^\alpha}\Big).$$

It is trivial that $0 < K \le 1$. By an easy computation we get

$$|h_\alpha(a) - h_\alpha(b)| \le K \sum_{n>k} \frac{2}{n^\alpha}.$$

If $\alpha > 1$, then the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges, and therefore, $\sum_{n>k} \frac{1}{n^\alpha} \to 0$ for $k \to \infty$.

The proof is finished. □

Our aim will now be to investigate the closure of the set

$$M_\alpha = \{\log h_\alpha(s)\,; \ \ s \in S\} \tag{9}$$

where S is a subset of $\mathbb{N}$ having the measure density 1.

We shall prove the following theorem:

THEOREM 3. *For every $\alpha > 1$ there exists a finite number of closed intervals $[l_1 - C, l_1], \dots, [l_r - C, l_r]$, $C > 0$, such that for each set S having the measure density 1, the closure of the set M_α, given by (9), can be expressed in the form*

$$\overline{M}_\alpha = \bigcup_{i=1}^{r} [l_i - C, l_i].$$

Proof. The arithmetic function h_α can be extended to a continuous function

$$h_\alpha : \Omega \to \Big[\frac{1}{\zeta(\alpha)}, 1\Big]$$

by putting

$$h_\alpha(\gamma) = \prod_{p/\gamma} \Big(1 - \frac{1}{p^\alpha}\Big), \qquad \gamma \in \Omega.$$

From this fact we obtain that the function $\log h_\alpha$ is also continuous. Proposition 1 gives now

$$\overline{M}_\alpha = \log h_\alpha(\Omega).$$

□

The assertion now follows from the following properties of infinite series. Put for the sequence of real numbers $\{a_n\}$

$$V(\{a_n\}) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n a_n;\ \varepsilon_n = 0, 1 \right\}.$$

Lemma 2. *Let* $a_n > 0$, $n = 1, 2, \dots$ *and* $A = \sum_{n=1}^{\infty} a_n < \infty$.
If for every k *holds*

$$a_k \le \sum_{n=k+1}^{\infty} a_n,$$

then

$$V(\{a_n\}) = [0, A].$$

Proof. See [10, p. 95]. □

Lemma 3. *Let* $\{a_n\}$ *be a decreasing sequence of positive real numbers such that* $\sum_{n=1}^{\infty} a_n < \infty$ *and*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

Then there exists $C > 0$ *and a finite number of closed intervals*

$$[l_1, l_1 + C], \dots [l_r, l_r + C], \quad l_i \ge 0, \ (i = 1, 2, \dots, r)$$

such that

$$V(\{a_n\}) = \bigcup_{j=1}^{r} [l_j, l_j + C].$$

Proof. The assertion is a trivial consequence of Lemma 2, taking into account that there exists k_0 such that $\frac{a_k}{a_{k+2}} \le 2$ for $k \ge k_0$. In this case

$$\sum_{n=k+1}^{\infty} a_n \ge a_{k+1} + a_{k+2} \ge 2a_{k+2} \ge a_k.$$

If $C = \sum\limits_{k=k_0}^{\infty} a_k$ and $\{l_1, \dots, l_r\}$ will be the set of all numbers in the form $\varepsilon_1 a_1 + \dots + \varepsilon_{k_0} a_{k_0}$, $\varepsilon_i = 0, 1$ the assertion follows. $\square$

Now we turn back to the proof of Theorem 3.

Let $p_1 < p_2 < \dots$ be the increasing sequence of all primes. It is trivial that $\log\, h_\alpha(\Omega) = V(\{a_n\})$, where $a_n = \log(1 - \frac{1}{p_n{}^\alpha})$, $n = 1, 2, \dots$ Then $-\log\, h_\alpha(\Omega) = V(\{-a_n\})$. The sequence $\{-a_n\}$ is decreasing and $\sum\limits_{n=1}^{\infty} a_n = \log \zeta(\alpha) < \infty$, for $\alpha > 1$. Clearly $\lim\limits_{n \to \infty} \frac{a_n}{a_{n+1}} = 1$. Finally from Lemma 3 we obtain the assertion of Theorem 3.

Let $\{x_n\}$ be a sequence of positive integer numbers. The sequence $\{x_n\}$ is said to be *uniformly distributed* in the set of integers if and only if for every positive integers j and m it holds that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{x_n \equiv j(m) \\ n \le N}} 1 = \frac{1}{m}\,.$$

Let us remark that $\mu^*(S) = 1$ if and only if S can be rearranged into a sequence which is uniformly distributed in $\mathbb{Z}$ (cf. [7, p. 27]).

In the monograph [3, p. 41], it is proved that $\{x_n\}$ is uniformly distributed in the set of all integers if and only if for every positive integer m and every periodic arithmetic function g with period m, the following is valid: $\lim\limits_{N \to \infty} \frac{1}{N} \sum\limits_{n-1}^{N} g(x_n) = \frac{1}{m} \sum\limits_{j=0}^{m-1} g(j)$. The following simple consideration shows us that this criterion can be extended to a larger class of functions.

In [9, p. 201] it is proved that for each polyadic continuous arithmetic function there exists the limit

$$S_f = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n)\,. \tag{10}$$

Clearly for each polyadic continuous function f there exists a periodic g such that

$$\big|g(n) - f(n)\big| < \varepsilon, \quad n = 1, 2, \dots\,.$$

This immediately implies:

PROPOSITION 2. *The sequence of positive integers $\{x_n\}$ is uniformly distributed in the set of all integers if and only if for every polyadically continuous* *arithmetic function* f *it holds*

$$S_f = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n),$$

where S_f *is given by (10).*

As a consequence, we now prove one equality for sequences that are uniformly distributed in the set of all integers.

THEOREM 4. *If* $\{x_n\}$ *is a sequence of positive integers which is uniformly distributed in the set of all integers, then for each* $\alpha > 1$ *it holds*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h_\alpha(x_n) = \frac{1}{\zeta(\alpha+1)}.$$

For proof we shall need the following assertion well known as Wintner's theorem:

THEOREM C. *Let* f *be an arithmetic function with representation*

$$f(n) = \sum_{d|n} F(d)$$

and for the function F *we have*

$$\sum_{d=1}^{\infty} \frac{|F(d)|}{d} < \infty,$$

then S_f *exists and*

$$S_f = \sum_{d=1}^{\infty} \frac{F(d)}{d}.$$

Proof. See [9, p. 192]. □

Proof of Theorem 4. According to Proposition 2 and Lemma 1 for each $\alpha > 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h_\alpha(x_n) = S_{h_\alpha}.$$

Let μ denote the Möbius function. Clearly we have

$$h_\alpha(n) = \sum_{d/n} \frac{\mu(d)}{d^\alpha}, \quad n = 1, 2, \ldots$$

Therefore, from Wintner's theorem we obtain

$$S_{h_\alpha} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{\alpha+1}}.$$

Using (4) we get

$$S_{h_\alpha} = \frac{1}{\zeta(\alpha+1)}.$$

Thus, the proof is complete. □

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