Tatra Mountains Math. Publ., **2** (1993), 203–207

TATRA MOUNTAINS
Mathematical Publications

ATOMIC ORTHOPOSETS WITH ORTHOMODULAR MACNEILLE COMPLETIONS

ZDENKA RIEČANOVÁ

Dedicated to the memory of Tibor Neubrunn

ABSTRACT. In this note we find a necessary and sufficient condition under which the MacNeille completion of an atomic orthoposet is a complete orthomodular lattice; as a partial answer to the question of J. Tkadlec [11].

1. Introduction and basic notions

A structure $(P, \leq, \perp, 0, 1)$ is called an *orthoposet* (we shall shortly denote it by P) if $(P, \leq)$ is a poset and the unary operation $\perp : a \in P \to a^{\perp} \in P$ is such that for every $x, y \in P$

(i) $(x^{\perp})^{\perp} = x$,

(ii) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$,

(iii) $x \vee x^{\perp} = 1$,

(iv) $0 = 1^{\perp}$.

An orthoposet P is called an *orthomodular poset* if $x \vee y$ exists for any pair $x, y \in P$ such that $x \leq y^{\perp}$, and the orthomodular law is valid in P, i.e. $y = x \vee (x^{\perp} \wedge y)$ for every $x, y \in P$ such that $x \leq y$. An orthomodular poset which is a lattice is called an *orthomodular lattice.* A maximal Boolean subalgebra of an orthomodular lattice is called a *block.*

We say that elements x, y of an orthoposet P are *orthogonal* if $x \leq y^{\perp}$. The set $M \subseteq P$ is said to be orthogonal if every two different elements of M are orthogonal.

A nonzero element a of an orthoposet P is an *atom* if $b \leq a \implies b = 0$ or $b = a$. An orthoposet P is *atomic* if every nonzero element of P contains

AMS Subject Classification (1991): 06C15, 81P10.
Key words: orthomodular poset, orthomodular lattice, Mac Neille completion, atom, atomic, atomistic.

an atom, and P is *atomistic* if every $x \in P$ is the supremum of all atoms lying under it. Note, that an atomic orthoposet need not be atomistic, while every nonzero element of an atomic orthomodular lattice is the supremum of an orthogonal set of atoms (see [4, p. 140]). Two elements x, y of an orthomodular poset P are called *compatible* if there exist mutually orthogonal elements $a_1, c_1, b \in P$ such that $x = c_1 \vee b$ and $y = a_1 \vee b$. The set of all the elements $x \in P$ which are compatible with every element of P is called the *center* of P.

LEMMA 1. *Let P be an atomistic orthocomplete orthomodular poset. Then:*

(i) *$C(P) = \{x \in P \mid a \leq x$ or $a \leq x^{\perp}$ for every atom a of $P\}$ is the center of P.*

(ii) *Every element $x \in P$ is the supremum of an orthogonal set of atoms of P.*

(iii) *To every $x, y \in P$ such that $x \leq y$ there are orthogonal sets A_x, A_y of atoms of P with $A_x \subseteq A_y$ and $\bigvee A_x = x$, $\bigvee A_y = y$.*

Proof. (i) Let us denote $C^*(P)$ the center of P.

Let $x \in C^*(P)$. For every atom a of P we have $x = a_1 \vee b$, $a = c_1 \vee b$, where a_1, c_1, b are mutually orthogonal elements of P. Now, if $b = a$ then $a \leq x$, if $b = 0$ then $x = a_1$ is orthogonal to a, hence $a \leq x^{\perp}$. We obtain $x \in C(P)$.

Let $x \in C(P)$. If $a \in A(P)$ such that $a \leq x$ then 0, a, $a^{\perp} \wedge x$ are mutually orthogonal elements with $x = a \vee (a^{\perp} \wedge x)$, $a = a \vee 0$. If $a \in A(P)$ and $a \leq x^{\perp}$ then 0, a, x are mutually orthogonal elements with $x = 0 \vee x$, $a = 0 \vee a$. Thus $x \in C^*(P)$.

(ii) It follows from the orthocompleteness of P making use the Zorn's Lemma and the orthomodular law.

(iii) We put $A_y = A_x \bigcup A_{x^{\perp} \wedge y}$, where $A_x, A_{x^{\perp} \wedge y}$ are orthogonal sets of atoms of P such that $x = \bigvee A_x$ and $x^{\perp} \wedge y = \bigvee A_{x^{\perp} \wedge y}$. □

2. MacNeille completions of atomic orthoposets

It is well known that any partially ordered set P can be embedded into its MacNeille completion $\overline{P}$ (or completion by cuts). It has been shown (see [9]) that any complete lattice $\overline{P}$ into which P can be supremum-densely and infimum-densely embedded (i.e. every element of $\overline{P}$ is the supremum of elements of the image of P and the infimum of elements of the image of P) is isomorphic to the MacNeille completion of P. For an orthoposet P the MacNeille completion is always a complete ortholattice (see [4, pp. 255–256]) in which orthocomplementation extends that of P. If P is an atomistic orthoposet then the previous observations imply the following lemma.

LEMMA 2. *Let P be an atomistic orthoposet and let $\overline{P}$ be its MacNeille completion. Then the set of all atoms of $\overline{P}$ coincides with the set of all atoms of $\varphi(P)$, where $\varphi\colon P \to \overline{P}$ is the embedding. Moreover P is atomic (atomistic, resp.) iff $\overline{P}$ is atomic (atomistic, resp.).*

Proof. By [4], $\overline{P}$ is a complete ortholattice. We can identify P with $\varphi(P)$, where $\varphi\colon P \to \bar{P}$ is the embedding. Then P is supremum dense in $\overline{P}$ and thus an element $a \in \overline{P}$ is an atom of $\overline{P}$ if and only if a is an atom of P. It follows the statements. □

It is known that the MacNeille completion of an orthomodular lattice P is not necessarily orthomodular, even if P is modular (see [1]). Positive results have been obtained in [3 and 5–8].

THEOREM 1. *Let P be an atomic orthoposet and let $A(P) = \{a \in P \mid a$ is an atom of $P\}$. The following conditions are equivalent:*

(i) *The MacNeille completion $\overline{P}$ of P is a complete orthomodular lattice.*
(ii) *P is atomistic and for any $S_1 \subseteq S_2 \subseteq A(P)$ there exist $D_1 \subseteq D_2 \subseteq A(P)$ with D_1, D_2 orthogonal and $\bigvee D_1 = \bigvee S_1$, $\bigvee D_2 = \bigvee S_2$ (in $\overline{P}$).*
(iii) *P is atomistic and for any $S_1 \subseteq S_2 \subseteq A(P)$ there exist $D_1 \subseteq D_2 \subseteq A(P)$ with D_1, D_2 orthogonal and $\bar{D}_1 = \bar{S}_1$, $\bar{D}_2 = \bar{S}_2$; where for every $S \subseteq A(P)$ we have $\bar{S} = \{p \in A(P) \mid a \in A(P), a \leq b^{\perp}$ for every $b \in S$ implies $a \leq p^{\perp}\}$.*

Proof. (i)$\Rightarrow$(ii): In view of Lemma 2, $\overline{P}$ is atomic and thus by [4, p. 140] we obtain that $\overline{P}$ is atomistic. The rest is a direct consequence of Lemma 1, (iii) and Lemma 2.

(ii)$\Rightarrow$(i): Suppose that x, y are elements of the MacNeille completion $\overline{P}$ of P such that $x \leq y$. Let $A_x = \{a \in A(P) \mid a \leq x\}$, $A_y = \{a \in A(P) \mid a \leq y\}$, (we identify P with $\varphi(P)$, where $\varphi\colon P \to \overline{P}$ is the embedding). Evidently $A_x \subseteq A_y$. In view of the assumption (ii) there exist orthogonal sets of atoms $D_x \subseteq D_y \subseteq A(P)$ such that $\bigvee D_x = \bigvee A_x = x$, $\bigvee D_y = \bigvee A_y = y$. It follows that $y = \bigvee D_x \vee \bigvee(D_y \setminus D_x) \leq x \vee (x^{\perp} \wedge y) \leq y$. Hence $\overline{P}$ is an orthomodular lattice.

(ii)$\Leftrightarrow$(iii): For every $S \subseteq A(P)$ we put $S^* = \{p \in A(P) \mid p \leq \bigvee S$ (in $\overline{P}$)$\}$, we identify P with $\varphi(P)$, where $\varphi\colon P \to \overline{P}$ is the embedding). In view of Lemma 2, we have $S^* = \bar{S} = \{p \in A(P) \mid a \in A(P), a \leq b^{\perp}$ for every $b \in S$ implies $a \leq p^{\perp}\}$. It follows that (ii) and (iii) are equivalent. □

THEOREM 2. *Let P be an atomic orthoposet and $\overline{P}$ be the MacNeille completion of P. Let $A(P) = \{a \in P \mid a$ is an atom of $P\}$ and $\mathcal{A} = \{D \subseteq A(P) \mid D$ is a maximal orthogonal set of atoms of $P\}$. If one of the equivalent conditions of Theorem 1 is fulfilled then:*

(i) *To every orthogonal set $F \subseteq A(P)$ there exist $D \in \mathcal{A}$ and a block B of $\overline{P}$ such that $F \subseteq D \subseteq B$ and B is isomorphic to the power set of D (we identify P with $\varphi(\overline{P})$, where $\varphi\colon P \to \overline{P}$ is the embedding).*

(ii) *$\overline{P} = \bigcup\{B \subseteq \overline{P} \mid B$ is a block of $\overline{P}$ with $B \supseteq D$ for some $D \in \mathcal{A}\}$.*

(iii) *$C(P) \subseteq C(\overline{P})$, where $C(\overline{P})$ is the center of $\overline{P}$ and $\overline{C(P)}$ is the MacNeille completion of $C(P) = \{x \in P \mid$ for every $a \in A(P)$ either $a \leq x$ or $a \leq x^{\perp}\}$.*

Proof. (i) Let F be an orthogonal set of atoms of P and $d = \bigvee F$ (in $\overline{P}$). Put $A_{d^{\perp}} = \{a \in A(P) \mid a \leq d^{\perp}\}$. Let $A^*_{d^{\perp}}$ be an orthogonal set of atoms with $\bigvee A^*_{d^{\perp}} = \bigvee A_{d^{\perp}}$ (in $\overline{P}$). In view of $d \vee d^{\perp} = 1$ the set $D = F \bigcup A^*_{d^{\perp}}$ is a maximal orthogonal set of atoms of P. In the opposite case there exists $q \in A(P)$, q orthogonal to every $p \in D$. Then $q \not\leq \bigvee D = 1$ is a contradiction. Let B be a block of $\overline{P}$ such that $D \subseteq B$. Then B is a complete Boolean algebra with D as a set of all atoms of B. Thus B is the power set of D.

(ii) Let $x \in \overline{P}$. Since $\overline{P}$ is an orthomodular lattice, there exists an orthogonal set $F_x \subseteq A(P)$ with $\bigvee F_x = x$. In view of (i). there exist $D \in \mathcal{A}$ and a block B of $\overline{P}$ such that $F_x \subseteq D \subseteq B$. Evidently $x \in B$.

(iii) In view of Lemma 1 and 2 we obtain $C(P) \subseteq C(\overline{P})$. Let $M \subseteq C(P)$ and $x = \bigvee M$ (in $\overline{P}$). For every $a \in A(P)$ we have either $a \leq y$ for some $y \in M$ and hence $a \leq x$, or $a \leq y^{\perp}$ for every $y \in M$ and hence $a \leq x^{\perp}$. By Lemma 1 we obtain that $x \in C(\overline{P})$. Since for every $y \in \overline{C(P)}$ there exists $K \subseteq C(P)$ such that $y = \bigvee K$ we conclude that $\overline{C(P)} \subseteq C(\overline{P})$. □

Recall that an orthoposet P is called *Boolean* if $x \wedge y = 0$ implies $x \leq y^{\perp}$, for every $x, y \in P$. Suppose that P is an atomic Boolean orthoposet. Evidently every two different atoms of P are orthogonal. Let $x, y \in P$ be such that $a \leq x$ implies $a \leq y$ for every atom a of P. Then $x \leq y$ (since $x \not\leq y$ implies $x \wedge y^{\perp} \neq 0$ and hence there exists atom a of P such that $a \leq x \wedge y^{\perp}$, a contradiction) and thus P is atomistic. We obtain the following corollary of Theorem 1 and 2.

COROLLARY 1. *The MacNeille completion $\overline{P}$ of an atomic Boolean orthoposet P is a Boolean algebra isomorphic to the power set of the set of all atoms* of P.

REFERENCES

[1] ADAMS, D. H.: *The completion by cuts of an orthocomplemented modular lattice*, Bull. Austral. Math. Soc. **1** (1969), 279–280.

[2] BRUNS, G.—GREECHIE, R. J.—HARDING, J.—RODDY, M.,: *Completions of orthomodular lattices*, Order 7 (1990), 67–76.

[3] JANOWITZ, M. F.: *Indexed orthomodular lattices*, Math. Z. **119** (1971), 28–32.

[4] KALMBACH, G.: *Orthomodular Lattices*, Academic Press, London, 1983.

[5] PULMANNOVÁ, S.—RIEČANOVÁ, Z.: *Compact topological orthomodular lattices*, Contributions to general algebra **7** (1991), 277–283, Verlag Hölder–Pichler–Tempsky, Wien.

[6] PULMANNOVÁ, S.—ROGALEWICZ, V.: *Orthomodular lattices with almost orthogonal sets of atoms*, Comment. Math. Univ. Carolinae **32, 3** (1991), 423–429.

[7] RIEČANOVÁ, Z.: *Applications of topological methods to the completion of atomic orthomodular lattices*, Demostratio Mathematica **XXIV** No. 1–2 (1991), 332–339.

[8] RIEČANOVÁ, Z.: *Topological and order-topological orthomodular lattices*, Bull. Austral. Math. Soc. **47** (1993), 509–518.

[9] SCHMIDT, J.: *Zur Kennzeichnung der Dedekind-Mac Neilleschen Hulle einer Geordneten Menge*, Arch. Math. **7** (1956), 241–249.

[10] TAE HO CHOE—GREECHIE, R., J.: *Profinite orthomodular lattices*, PAMS (to appear).

[11] TKADLEC, J.: *Partially additive measures and set representations of orthoposets*, J. Pure Appl. Algebra (to appear).

Received October 29, 1992

Department of Mathematics
Faculty of Electrical Engineering
Slovak Technical University
Ilkovičova 3
812 19 Bratislava
SLOVAKIA