Tatra Mountains Math. Publ., **2** (1993), 209–219

TATRA MOUNTAINS
Mathematical Publications

A REMARK ON STATES ON ORTHOMODULAR LATTICES

SYLVIA PULMANNOVÁ

Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Some simple properties of states and state spaces on orthocomplemented structures are studied. Relations between state spaces and some structural properties are found.

In this paper, we will study some simple but useful properties of states on ortholattices, orthomodular lattices, orthoposets and orthomodular posets. We will find some conditions under which these structures become Boolean algebras. Some of our statements (and also some more) can be found in existing literature, nevertheless also some original results are proved. Our main sources are [2, 8, 11, 12, 20, 28]. An ortholattice (OL) L is a lattice with 0 and 1 endowed with a unary operation $' : L \to L$ (orthocomplementation) such that

(i) $a \le b \Rightarrow b' \le a'$; (ii) $a'' = a$;
(iii) $a \vee a' = 1$ $(a \wedge a' = 0)$.

If, in addition, we have

(iv) $a \le b \Rightarrow b = a \vee (a' \wedge b)$ (orthomodular law),

then L is an orthomodular lattice (OML).

A *state* on an OL L is a mapping $m : L \to [0,1]$ such that

(i) $m(1) = 1$,
(ii) we say that a, b in L are orthogonal (written $a \perp b$) if $a \le b'$, and $m(a \vee b) = m(a) + m(b)$ whenever a and b are orthogonal.

In the next definition, a classification of states is given

DEFINITION 1. Let L be an OL, m a state on L. We say that m is

(1) *Jauch–Piron* if

$$a, b \in L,\ m(a) = m(b) = 0 \Rightarrow m(a \vee b) = 0\,,$$

AMS Subject Classification (1991): 06A06, 06C15, 81P05, 81P15.
Key words: state, ortholattice, orthoposet, orthomodular lattice, orthomodular poset.

(2) a *(P)-state* if

$$a, b \in L,\ m(a) = 0 \Rightarrow m(a \vee b) = m(b)\,,$$

(3) a *(B)-state* if

$$a, b \in L,\ a \wedge b = 0 \implies m(a \vee b) = m(a) + m(b)\,,$$

(4) *subadditive* if

$$a, b \in L,\ m(a \vee b) \leq m(a) + m(b)\,,$$

(5) a *valuation* if

$$a, b \in L,\ m(a \vee b) + m(a \wedge b) = m(a) + m(b)\,.$$

We note that the notion of a Jauch–Piron state was introduced by Rüttimann [24] and properties of Jauch–Piron states were studied in [4, 5, 9, 10, 16, 17, 20]. A property similar to a (P)–state (in a stronger $\varepsilon-\delta$ form) was considered by von Neumann [18] and Dobrakov [6]. As concerns valuations, see, e.g., [3, 21, 22, 26].

In the next proposition, we collect some simple relations between different kinds of states on orthomodular lattices. If a, b are elements of an OML L and $a \leq b$, we will write $b - a$ for $b \wedge a'$.

PROPOSITION 2. *Let m be a state on an OML L. Let (1), (2), (3), (4), (5) be the properties of Definition 1. The following implications hold:* (5) $\iff$ (4) $\iff$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) *and* (2) $\not\Rightarrow$ (3), (1) $\not\Rightarrow$ (2).

Proof. (5) $\Rightarrow$ (4), (5) $\Rightarrow$ (3), (4) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are evident. (4) $\Rightarrow$ (3): Let $a \wedge b = 0$. Using subadditivity of m, we obtain

$$\begin{aligned} m(a \vee b) &= m((a \vee b) \wedge (a' \vee b')) = \\ &= m(a' \wedge (a \vee b) \vee (b' \wedge (a \vee b))) \leq \\ &\leq m((a \vee b) - a) + m((a \vee b) - b) = \\ &= m(a \vee b) - m(a) + m(a \vee b) - m(b)\,. \end{aligned}$$

Hence

$$m(a) + m(b) \leq m(a \vee b).$$

$(3) \Rightarrow (5)$: Let m be (B)–state, then, for any $a, b \in L$,

$$((a \vee b) \wedge a') \wedge ((a \vee b) \wedge b') = 0 \implies$$
$$m((a \vee b) \wedge (a \wedge b)') = m((a \vee b) \wedge a') \vee (a \vee b) \wedge b'))$$
$$= m((a \vee b) \wedge a') + m((a \vee b) \wedge b')) .$$

Hence

$$m(a \vee b) - m(a \wedge b) = m(a \vee b) - m(a) + m(a \vee b) - m(b),$$

which gives the desired result.

$(2) \not\Rightarrow (3)$: Let $L = L_1 \times L_2$ (where $\times$ denotes the direct product) and $L_1 =$ MO(2) be the horizontal sum of two Boolean algebras $\{0, 1, a, a'\}$, $\{0, 1, b, b'\}$, L_2 be an arbitrary OML. Let m_1 be the state on L_1 defined by $m_1(0) = 0$, $m_1(1) = 1$, $m_1(a) = 1/3$, $m_1(b) = 1/2$. Define a state m on L by $m((u, v)) = m_1(u)$. Let (t, x), (y, z) be such that $m((t, x)) = 0$. Then $t = 0$, hence

$$m((t, x) \vee (y, z)) = m((y, x \vee z)) = m_1(y) = m((y, z))$$

and hence m is a (P)–state. But m is not a (B)–state, since $(a, 0) \wedge (b, 0) = 0$, but $m((a, 0) \vee (b, 0)) = m((1, 0)) = 1 \neq m((a, 0)) + m((b, 0))$.

$(1) \not\Rightarrow (2)$: Let $L = MO(2) = \{0, 1, a, a', b, b'\}$ and define a state m on L by putting $m(a) = 0, m(b) = 1/2$. Then m is Jauch–Piron, but we have $m(a) = 0, m(a \vee b) = m(1) = 1 \neq m(b)$, hence m is not a (P)–state. □

For a state m on an OL L, the implications $(5) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$ and $(5) \Rightarrow (3)$ remain true.

A state m on L is a $\{0, 1\}$–state if $m(a) \in \{0, 1\}$ for every $a \in L$. It is easy to check that the following is true.

PROPOSITION 3. *Let m be a $\{0, 1\}$–state on an OL L. Then m is a valuation iff m is Jauch–Piron.*

Recall that a subset I of an OML L is

(i) an order ideal if

$$a \in I, \ b \in L, \ b \leq a \implies b \in I,$$

(ii) an ideal if I is an order ideal and

$$a \in I, \ b \in I \implies a \vee b \in I,$$

(iii) a p-ideal if I is an ideal and

$$a \in I, \ b \in L \implies (a \vee b') \wedge b \in I.$$

For a state m on L, put $\mathcal{N}(m) = \{a \in L \colon m(a) = 0\}$.

PROPOSITION 4. *Let L be an OML and m be a state on L.*

(i) *$\mathcal{N}(m)$ is an order ideal in L and $a, b \in \mathcal{N}(m)$, $a \perp b \implies a \vee b \in \mathcal{N}(m)$.*

(ii) *$\mathcal{N}(m)$ is an ideal iff m is Jauch–Piron.*

(iii) *$\mathcal{N}(m)$ is a p-ideal iff m is a (P)–state* (see also [1]).

Proof. Is left to the readers as an easy exercise. □

Remark 5. If m is a valuation, then $\mathcal{N}(m)$ is a p-ideal, the converse need not hold (just consider a faithful state m (i.e., a state with $\mathcal{N}(m) = \{0\}$), not necessarily a valuation).

DEFINITION 6. Let S be a set of states on an OML L. We say that S is

(i) *unital* if

$$a \in L,\ a \neq 0 \implies \exists\, s \in S\colon s(a) = 1,$$

(ii) *full* (or *ordering*) if

$$a, b \in L,\ a \not\leq b \implies \exists\, s \in S\colon s(a) \gneq s(b),$$

(iii) *rich* (or *strongly ordering*) if

$$a, b \in L,\ a \not\leq b \implies \exists\, s \in S\colon s(a) = 1,\ s(b) \neq 1.$$

(see [2, 8, 11, 12, 20]).

It is easy to see that a rich set S of states on an OL L is both unital and full; the converse implication need not hold. Also, there is no relation between unital and full sets, in general. On the other hand, it is easy to see that an ordering set of $\{0,1\}$–states is rich.

PROPOSITION 7. *A unital set of Jauch–Piron states on an OML L is rich.*

Proof. Let $a \not\leq b$. Then $a \wedge b \lneq a$, hence $a \wedge (a \wedge b)' \neq 0$ (by orthomodularity). Hence there is a state s such that $s(a \wedge (a \wedge b)') = 1$. This implies $s(a) = 1$, $s(a \wedge b) = 0$. Since s is Jauch–Piron, we have $s(b) \neq 1$. □

Now assume that L is an ortholattice (OL), not necessarily orthomodular. Recall that, for every $x \in L$, the Sasaki projection $\phi_x : L \to L$ is defined by $\phi_x(y) = x \wedge (x' \vee y)$. It is a well-known fact that an ortholattice L is orthomodular iff

$$a, b \in L,\ a \leq b \implies \phi_b(a) = a,$$

and an ortholattice L is a Boolean algebra iff

$$a, b \in L \Rightarrow \phi_b(a) = a \wedge b.$$

PROPOSITION 8. *Let L be an OL and m a state on L. Then $\mathcal{N}(m)$ is an order ideal such that $a, b \in \mathcal{N}(m)$, $a \perp b \implies a \vee b \in \mathcal{N}(m)$ and*

$$a, b \in L,\ a \in \mathcal{N}(m),\ a \leq b \implies \phi_b(a) \in \mathcal{N}(m)\,. \tag{a}$$

If m is a (P)–state, then (a) can be replaced by

$$a, b \in L,\ a \in \mathcal{N}(m) \implies \phi_b(a) \in \mathcal{N}(m)\,. \tag{b}$$

Proof. For $a, b \in L$, $a \leq b$ implies $a \perp b'$. Therefore $m(a \vee b') = m(a) + 1 - m(b) \implies m(b) = m(a) + 1 - m(a \vee b') = m(a) + m(a' \wedge b)$. Now $m(\phi_b(a)') = m(b' \vee (b \wedge a')) = m(b') + m(b \wedge a')$. If $m(a) = 0$, then $m(\phi_b(a)') = m(b') + m(b) - m(a) = 1$, hence $\phi_b(a) \in \mathcal{N}(m)$. If m is a (P)–state then $m(a) = 0 \implies 0 = m(a \vee b) - m(b) = m(\phi_b(a))$. □

Remark 9. An OL L is an OML iff every proper order ideal I has property (a) (with I instead of $\mathcal{N}(m)$); and an OL L is a Boolean algebra iff every proper order ideal I has property (b) (with I instead of $\mathcal{N}(m)$). It follows from the fact that every interval $[0, a], a \neq 1$ is a proper order ideal.

THEOREM 10. *Let L be an OL.*

(i) L is an OML iff

$$a, b \in L,\ a \not\leq b \implies \exists\, I,\ b \in I,\ a \notin I,$$

where I is a proper order ideal satisfying condition (a).

(ii) L is a Boolean algebra iff

$$a, b \in L,\ a \not\leq b \implies \exists\, I,\ b \in I,\ a \notin I,$$

where I is an order ideal satisfying condition (b).

Proof. (i) If L is an OML, then for $a, b \in L$, $a \not\leq b$, the interval $[0, b]$ satisfies the required condition.

Conversely, let $a \not\leq b$. Let I_b be the smallest order ideal with property (a) containing b. Then for every such order ideal I, $I \supset I_b$. This implies that $a \not\leq b \implies a \notin I_b$, hence $I_b = [0, b]$. But then for every $b \in L$, $b \neq 1$, the interval $[0, b]$ has property (a), hence L is an OML.

(ii) If L is a Boolean algebra, then for $a, b \in L$, $a \not\leq b$, the interval $[0, b]$ satisfies the required conditions.

Conversely, similarly as in case (i) we prove that every interval $[0, b]$ has property (b). Therefore L is a Boolean algebra. □

Corollary 11. *An OL L is*

(*i*) *an OML if it possesses a rich set of states;*

(*ii*) *a Boolean algebra iff it possesses a rich set of (P)–states.*

Proof. It follows from the properties of the sets $\mathcal{N}(m)$.

We note that we cannot write iff in the statement (i): Greechie [7] found examples of OMLs without states. On the other hand, the existence of a full set of Jauch–Piron $\{0,1\}$–states is necessary and sufficient to make an OL a Boolean algebra.

Theorem 12. *Let L be an OL.*

(*i*) *If L possesses a full set of states, then L is an OML.*

(*ii*) *L is a Boolean algebra iff it possesses a full set of valuations.*

Proof. (i) is left to the reader.

(ii) (see also [21]). We know from (i) that L is an OML. For any $a, b \in L$, take into account that $\phi_b(a)$ is perspective with $\phi_a(b)$, and that a valuation is equal on perspective elements (see, e.g., [11] for the definition of perspectivity). This gives that $m(\phi_b(a)) \leq m(a)$ for every valuation m, and since we have a full set of valuation, we get $\phi_b(a) = a \wedge b$, hence L is a Boolean algebra. □

Theorem 13. (See also [25]). *Let L be an OML. The following statements are equivalent*

(*i*) *L is a Boolean algebra;*

(*ii*) *L has a unital set of (P)–states;*

(*iii*) *for every $a \in L, a \neq 1$, there is a proper p-ideal I in L such that $a \in I$.*

Proof. (*i*) $\Longrightarrow$ (*ii*) is straightforward. (*ii*) $\Longrightarrow$ (*iii*) follows from the fact that for a (P)–state m, $\mathcal{N}(m)$ is a p-ideal. (*iii*) $\Longrightarrow$ (*i*) We will prove that condition (ii) in Theorem 10 is satisfied. Let $a, b \in L$, $b \not\leq a$. Then $b \vee a \gneq a$ implies that $a \vee (a' \wedge b') \neq 1$. Therefore there is a proper p-ideal I such that $a \vee (a' \wedge b') \in I$. Then $a \in I$ and if also $b \in I$, then $\phi_{a'}(b) = a' \wedge (a \vee b) \in I$, which together with $a \vee (a' \wedge b') \in I$, contradicts the condition that I is proper. Hence $b \notin I$. □

The remaining part of our paper will be devoted to partially ordered sets, not necessarily lattices. We recall that a partially ordered set L is called an orthoposet (OP) if $0, 1 \in L$, and there is a unary operation $' : L \to L$ (an orthocomplementation) such that

(i) $a \leq b \Longrightarrow b' \leq a'$,

(ii) $a \vee a' = 1$;

(iii) $a'' = a$;

(iv) $a \leq b' \implies a \vee b$ exists in L.

If, in addition, the orthomodular law

(v) $a \leq b \implies b = a \vee (a' \wedge b)$

is satisfied, L is an orthomodular poset (OMP).

A state m on an OP L is a function $m\colon L \to [0,1]$ such that $m(1) = 1$ and $a \leq b' \implies m(a \vee b) = m(a) + m(b)$.

We note that if L is an OP and $a, b \in L$, $a \leq b$, then for any state m on L we have $m(b) = m(a) + m(a' \wedge b)$. Indeed, it follows from $a \leq b$ that $a \perp b'$, so that $a \vee b'$ exists in L and $m(a \vee b') = m(a) + m(b')$. It is easy to prove that an OP with full set of states is an OMP.

In analogy with the case of lattices, let us consider the following properties of states.

DEFINITION 14. Let m be a state on an OP L. We say that m is

(i) *Jauch–Piron* if

$$a, b \in L,\ m(a) = 0 = m(b) \implies \exists\, c \in L\colon c \geq a, b \text{ such that } m(c) = 0;$$

(ii) a *(P)-state* if

$$a, b \in L,\ m(a) = 0 \implies \exists\, c\colon c \geq a, b \text{ such that } m(c) = m(b);$$

(iii) a *(⊥)-state* if

$$a, b \in L,\ a \wedge b = 0,\ m(a) = 1 \implies m(b) = 0;$$

(iv) a *(⊥⊥)-state* if

$$a, b \in L,\ a \wedge b = 0 \implies m(a) + m(b) \leq 1;$$

(v) a *(B)-state* if

$$a, b \in L,\ a \wedge b = 0 \implies \exists\, c\colon c \geq a, b \text{ such that } m(c) = m(a) + m(b);$$

(vi) *subadditive*, if

$$a, b \in L \implies \exists\, c\colon c \geq a, b \text{ such that } m(c) \leq m(a) + m(b);$$

(vii) a *valuation*, if

$$a, b \in L \implies \exists\, c, d\colon\ c \geq a, b \geq d \text{ such that } m(c) + m(d) = m(a) + m(b).$$

Orthomodular posets with Jauch–Piron states were extensively studied ([4, 16, 17, 20]), for subadditive states see [14]. The other properties are new. The following statement is obvious.

PROPOSITION 15. *Let m be a state on an OP L. With the notation from Definition 14, the following implications hold:*

$$\begin{array}{ccccccc} (vii) & \Longrightarrow & (vi) & \Longrightarrow & (ii) & \Longrightarrow & (i) \\ \Downarrow & & & & \Downarrow & & \\ (v) & \Longrightarrow & (iv) & \Longrightarrow & (iii)\,. & & \end{array}$$

We say that a subset I of L is an ideal if I is an order ideal and for any elements a, b in I there is a $c \in L$ such that $a, b \leq c$ and $c \in I$. We note that if $a \leq b$ then the element $\phi_b(a)$ exists in L.

PROPOSITION 16. *Let m be a state on an OP L and $\mathcal{N}(m) = \{a \in L\colon m(a) = 0\}$. Then*

(i) $\mathcal{N}(m)$ is an order ideal with the property $a, b \in \mathcal{N}(m)$, $a \perp b \Longrightarrow a \vee b \in \mathcal{N}(m)$ and

$$a, b \in L\,,\ a \leq b\,,\ a \in \mathcal{N}(m) \Longrightarrow \phi_b(a) \in \mathcal{N}(m). \tag{A}$$

(ii) m is Jauch–Piron iff $\mathcal{N}(m)$ is an ideal with property (A).

(iii) m is a (P)–state iff $\mathcal{N}(m)$ is an ideal with the property

$$a, b \in L,\ a \in \mathcal{N}(m) \Longrightarrow \exists\, c\colon c \geq a, b \text{ such that } c \wedge b' \in \mathcal{N}(m)\,. \tag{B}$$

(iv) m is a $\perp$–state iff

$$a, b \in L,\ a \wedge b = 0\,,\ a' \in \mathcal{N}(m) \Longrightarrow b \in \mathcal{N}(m)\,.$$

P r o o f . Is left to the readers as an easy exercise. □

We say that an orthoposet L is Boolean if $a \wedge b = 0 \Longrightarrow a \leq b'$ (see [13, 27, 29]). We note that a Boolean ortholattice is a Boolean algebra (indeed, it is uniquely complemented, see e.g. [11]). It is easy to see that an orthoposet L with a rich set of $\perp$–states or with a full set of $\perp\perp$–states is a Boolean orthomodular poset. It would be interesting to study which kinds of state spaces imply the lattice structure, resp. a structure of a Boolean algebra on an OP L. For example, it was proved in [23] that a block-finite OMP L, the state space of which is unital and consists of Jauch–Piron states, is a Boolean algebra. We obtain a similar result for (P)–states if we impose a finiteness condition on the set of states.

PROPOSITION 17. *(i) If an OP L has a finite rich set of (P)–states, it is a finite Boolean algebra.*

(ii) If an OML L has a finite unital set of (P)–states, it is a finite Boolean algebra.

Proof. (i) Let S denote the rich set of (P)–states on L. Our assumptions imply that L is an OMP. Let $a \in L$ and let there be $0 \neq a_1 \lneq a$. Then there is $s_1 \in S$ with $s_1(a) = 1$, $s_1(a_1) \neq 1$. If there is $0 \neq a_2 \lneq a_1$, then there is $s_2 \in S$ with $s_2(a_1) = 1$, $s_2(a_2) \neq 1$. Clearly, $s_1 \neq s_2$. Proceeding by induction, we show that there is an atom b under a. Hence L is atomic. Moreover, since a (P)–state is also a $\perp$-state (Proposition 15), and $a \wedge b = 0$ for any two different atoms a, b, we have $s(a) = 1 \implies s(b) = 0$ for any $s \in S$, hence $a \perp b$. This implies that the set of atoms is finite and any two atoms are orthogonal. From this we easily derive that L is a finite Boolean algebra.

(ii) follows from the fact that a unital set of (P)–states is rich for an OML. □

In the end, we introduce an example of an OMP with a valuation. Let H be a Hilbert space, $\mathcal{B}(H)$ the algebra of all bounded operators on H and $\mathcal{P} = \{P \in \mathcal{B}(H) : P^2 = P\}$. Elements of $\mathcal{P}$ are called skew projections. It was proved in [15] that, if $\dim H \geq 3$, $\mathcal{P}$ is an OMP which is not a lattice. The partial order on $\mathcal{P}$ is defined by $P \leq Q \Leftrightarrow PQ = QP = P \Leftrightarrow \mathcal{M}_P \subseteq \mathcal{M}_Q$ and $\mathcal{N}_P \supseteq \mathcal{N}_Q$, where $\mathcal{M}_P = \{Px : x \in H\}$ is the range of P and $\mathcal{N}_P = \{x \in H : Px = 0\}$ is the null space of P. Orthocomplementation is defined by $P' = I - P$. The identity I and the null projection 0 are the greatest and smallest element in $\mathcal{P}$, respectively. We have $P \perp Q$ iff $P \leq Q'$ iff $PQ = QP = 0$, therefore $P + Q$ is a skew projection. Moreover, $(P+Q)x = Px + Qx \in \mathcal{M}_P + \mathcal{M}_Q$, hence $\mathcal{M}_{P+Q} \subseteq \mathcal{M}_P + \mathcal{M}_Q$. Conversely, let $z \in \mathcal{M}_P + \mathcal{M}_Q$. Then $z = Px + Qy = Pz + Qz \in \mathcal{M}_{P+Q}$. Hence $\mathcal{M}_{P+Q} = \mathcal{M}_P + \mathcal{M}_Q$. Now $Px = Qx = 0 \implies (P+Q)x = 0$, hence $\mathcal{N}_P \wedge \mathcal{N}_Q \subseteq \mathcal{N}_{P+Q}$. Conversely, $x \in \mathcal{N}_{P+Q} \implies Px = -Qx$, and $0 = QPx = -Qx$, $0 = -PQx = Px$, hence $\mathcal{N}_{P+Q} = \mathcal{N}_P \wedge \mathcal{N}_Q$. This proves that $P + Q = P \vee Q$. From this we easily derive that $\mathcal{P}$ is an OMP.

Assume that $\dim H = 3$, and define $s \colon \mathcal{P} \to [0,1]$ by $s(P) = (1/3) \dim \mathcal{M}_P$. To prove additivity of s, let $P, Q \in \mathcal{P}$, $P \perp Q$. In this case, $P \vee Q = P + Q$, $\mathcal{M}_{P+Q} = \mathcal{M}_P + \mathcal{M}_Q$, and $\mathcal{M}_P \wedge \mathcal{M}_Q \subseteq \mathcal{M}_P \wedge \mathcal{N}_P = 0$. From this it follows that $s(P \vee Q) = s(P) + s(Q)$. Hence s is a state. Now take into account that if $\mathcal{M}$ and $\mathcal{N}$ are nontrivial complementary subspaces, then one of them is one-dimensional and the other is two-dimensional. Moreover, $P \vee Q = I$ if either $\mathcal{M}_P \vee \mathcal{M}_Q = H$ or $\mathcal{N}_P \wedge \mathcal{N}_Q = 0$ ($P, Q \in \mathcal{P}$). For example, let $\mathcal{M}_P = \mathcal{M}_Q$ be two-dimensional, $\mathcal{N}_P$ and $\mathcal{N}_Q$ be one-dimensional and different. Then $P \vee Q = I$ and $P \wedge Q$ does not exist, there are infinitely many lower bounds of P, Q: any projection R with $\mathcal{N}_R = \mathcal{N}_P \vee \mathcal{N}_Q$ and with $\mathcal{M}_R$ any one-dimensional subspace of $\mathcal{M}_P$ not lying in $\mathcal{N}_P \vee \mathcal{N}_Q$, is such a lower bound. With any such R we have $R \leq P, Q \leq I$

and $s(I)+s(R) = s(P)+s(Q)$. All other possibilities can be checked in a similar way. We note that our state s is the unique state on $\mathcal{P}$ (see [15]).

REFERENCES

[1] D'ANDREA, A. B.—PULMANNOVÁ, S.: *Boolean quotients of orthomodular lattices*, Preprint.

[2] BELTRAMETTI, E.—CASSINELLI, G.: *The Logic of Quantum Mechanics*, Addison--Wesley, Reading, Mass., 1981.

[3] BIRKHOFF, G.: *Lattice Theory*, AMS Colloq. Publ., New York, 1948.

[4] BUGAJSKA, K.—BUGAJSKI, S.: *On the axioms of quantum mechanics*, Bull. Acad. Polon. Sci., Sér. Math. **21** (1972), 231–234.

[5] BUNCE, L.—HAMHALTER, J.: *Jauch–Piron states on von Neumann algebras* (to appear).

[6] DOBRAKOV, I.: *On submeasures I*, Dissertationes Math. **112** (1974), 5–39.

[7] GREECHIE, R.: *Orthomodular lattices admitting no states*, J. Combin. Theory **10 A** (1971), 119–132.

[8] GUDDER, S.: *Stochastic Methods in Quantum Mechanics*, North–Holland, Amsterdam, 1979.

[9] HAMHALTER, J.: *Pure Jauch–Piron states on von Neumann algebras* (to appear).

[10] JAUCH, J.—PIRON, C.: *On the structure of quantal proposition systems*, Helv. Phys. Acta **42** (1969), 842–848.

[11] KALMBACH, G.: *Orthomodular Lattices*, Academic Press, London, 1983.

[12] KALMBACH, G.: *Measures and Hilbert Lattices*, World Scientific, Singapore, 1986.

[13] KLUKOWSKI, J.: *On Boolean orthomodular posets*, Demonstratio Math. **8** (1975), 5–14.

[14] DE LUCIA—P., PTÁK, P.: *Quantum probability spaces that are nearly classical* (to appear).

[15] MUCHTARI, D.—MATVEICHUK, M.: *Charges on the logic of skew projections*, Soviet math. Dokl. **32** (1985), 36–39.

[16] MÜLLER, V.: *Jauch–Piron states on concrete logics* (to appear).

[17] MÜLLER, V.—PTÁK, P.—TKADLEC, J.: *Concrete quantum logics with covering properties*, Internat. J. Theoret. Phys. **31** (1992), 843–854.

[18] vonNEUMANN, J.: *Continuous geometries with transition probability*, Mem. AMS **252** (1981) (Halperin, J., ed.).

[19] PIRON, C.: *Foundations of Quantum Physics*, Benjamin, Reading, Mass., 1976.

[20] PTÁK, P.—PULMANNOVÁ, S.: *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1991.

[21] PULMANNOVÁ, S.—MAJERNÍK, V.: *Bell inequalities on quantum logics*, J. Math. Phys. **33** (1992), 2173–2178.

[22] RIEČANOVÁ, Z.: *A topology on quantum logics induced by measures*, Proceedings of the Conference "Measure and Topology V.", Wiss. Beitrage EMA, Univ. Griefswald, Greifswald, 1988.

[23] ROGALEWICZ, V.: *Jauch–Piron logics with finiteness condition*, Internat. J. Theoret. Phys. **30** (1991), 437–445.

[24] RÜTTIMANN, G.: *Jauch–Piron states*, J. Math. Phys. **18** (1977), 189–193.

[25] SALVATI, S.: *Una caratterizazzione delle algebre di Boole tramite i p-ideali*, Preprint.

[26] SARYMSAKOV, T.—AJUPOV, S.—HADZIEV, D.—CHILIN, V.: *Ordered Algebras*, FAN, Taskent, 1983. (Russian)

[27] TKADLEC, J.: *Boolean orthoposets and two-valued states on them*, Preprint.

[28] VARADARAJAN, V.: *Geometry of Quantum Theory*, Springer, Berlin, 1985.

[29] NAVARA, M.—PTÁK, P.: *Almost Boolean orthomodular posets*, J. Pure Appl. Algebra **60** (1989), 105–111.

Received November 10, 1992

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
SLOVAKIA