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# THE CASIMIR CHAOS MAP FOR $U(N)$

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ABSTRACT. The generalized number processes of $N$-dimensional quantum stochastic calculus form a representation of the Lie algebra $\mathscr{L}$ of $U(N)$. Their stochastic differentials form a second such representation. Associated with each Casimir element $C$, we construct a *Casimir process* $(C_t\colon t \in \mathbb{R}_+)$ using the first representation, and a *Casimir chaos process* combining the second with an iterated integral which is defined naturally on the tensor algebra over $\mathscr{L}$ but is shown to extend to the center $\mathscr{Z}$ of the universal enveloping algebra in a natural way. The Casimir chaos process of $C$ is the Casimir process of the image of $C$ under a bijective linear map on $\mathscr{Z}$.

## 1. Introduction

In $N$-dimensional quantum stochastic calculus, in addition to creation and annihilation processes which do not concern us in this paper, there are generalized number processes [2, 5] $\bigl(\Lambda_H(t),\ t \geq 0\bigr)$ labelled by skew symmetric linear transformations $H$ on $\mathbb{C}^N$ satisfying

$$\bigl[\Lambda_H(t),\, \Lambda_K(t)\bigr] = \Lambda_{[H,K]}(t)\,.$$

Thus $H \mapsto \Lambda_H(t)$ is a representation of the Lie algebra of such skew symmetric transformations. In fact the $\Lambda_H(t)$ are operators of "differential second quantisation" and this is the infinitesimal representation of a representation $\pi_t$ of the group $U(N)$ of $N \times N$ unitary matrices got by a second quantisation procedure. More remarkably the Ito quantum stochastic differentials of the processes $\Lambda_H(t)$ also form a representation of the same Lie algebra; the quantum Ito formula [2, 5] gives

$$\bigl[d\Lambda_H(t),\, d\Lambda_K(t)\bigr] = d\Lambda_{[H,K]}(t)\,.$$

Our purpose in this paper is to exploit the existence of this linked pair of representations. The first representation extends in the usual way to the universal enveloping algebra and enables us in particular to associate a *Casimir process*

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$C(t)$ to each Casimir element, that is to each element $C$ of the center $\mathscr{Z}$ of the universal enveloping algebra $\mathscr{U}$. To exploit the second representation we consider iterated stochastic integrals. It emerges that such an integral is naturally associated with each element of $\mathscr{Z}$, though not of the universal enveloping algebra $\mathscr{U}$; thus we construct *Casimir chaos processes.* In addition products of such iterated integrals are again iterated integrals of the same type but the product of iterated integrals is not the iterated integral of the product, so that we have a representation of $\mathscr{Z}$ only if the multiplication is redefined. But it turns out that the totality of Casimir chaos processes is coextensive with the totality of Casimir processes obtained using the first representation. Thus there exists a *chaos map* $C \mapsto \tilde{C}$ in $\mathscr{Z}$ such that the Casimir process of $\tilde{C}$ gives the chaotic decomposition (that is, the expansion as an iterated integral) of the Casimir process of $C$.

## 2. Casimir elements

We identify the Lie algebra of the group $U(N)$ of $N \times N$ unitary matrices with the space of skew-symmetric linear operators on $\mathbb{C}^N$ in the natural way; then its complexification $\mathscr{L}$ is the space of all linear operators on $\mathbb{C}^N$ with the commutator Lie bracket and the natural involution $\dagger$. A basis for $\mathscr{L}$ is provided by the Dirac dyads $\Lambda^i_j$, $i, j = 1, \ldots, N$, which satisfy

$$[\Lambda^i_j, \Lambda^k_l] = \delta^k_j \Lambda^i_l - \delta^i_l \Lambda^k_j \,, \tag{2.1}$$

$$(\Lambda^i_j)^\dagger = \Lambda^j_i \,. \tag{2.2}$$

The adjoint action of $U$ extends to $\mathscr{L}$ as

$$\operatorname{ad} U(H) = UHU^{-1} \,. \tag{2.3}$$

To construct the universal enveloping algebra $\mathscr{U}$ [4] we first construct the free tensor algebra $\mathscr{I}$ generated by the linear space $\mathscr{L}$ equipped with the natural extension of the involution $\dagger$. We then take the quotient $\dagger$-algebra by the $\dagger$-ideal $\mathscr{J}$ generated by the set of elements of the form

$$H \otimes K - K \otimes H - [H, K], \quad H, K \in \mathscr{L} \,. \tag{2.4}$$

The adjoint action extends to the tensor algebra as

$$\operatorname{ad} U(H_1 \otimes \cdots \otimes H_n) = (UH_1U^{-1} \otimes \cdots \otimes UH_nU^{-1}) \,. \tag{2.5}$$

The ideal $\mathscr{J}$ is invariant under this action, thus we may pass to the quotient and allow that the adjoint action acts on $\mathscr{U}$. The center $\mathscr{Z}$ of $\mathscr{U}$ may then be characterized as the fixed point set of the adjoint action. Its elements are called Casimir elements. Every Casimir element has a unique representation, as an element of the quotient algebra $\mathscr{U} = \mathscr{I} / \mathscr{J}$, of form

$$C = c + \mathscr{J} \,, \tag{2.6}$$

where $c \in \mathscr{I}$ is a symmetric tensor. Since both $\mathscr{J}$ and the space of symmetric tensors are invariant under the adjoint action of $U(N)$ on $\mathscr{I}$, it is clear from the uniqueness that $c$ is pointwise invariant under this action. It is known [1] that $\mathscr{Z}$ is generated by the Casimir elements $D^{(r)}$, $r = 1, \ldots, N$ given by

$$D^{(r)} = \sum_{1 \leq i_1 < \cdots < i_r \leq N} \sum_{\sigma, \tau \in \mathscr{S}_r} \operatorname{sign}(\sigma\tau)\, \Lambda^{i(\sigma(1)}_{i_\tau(1)} \ldots \Lambda^{i_\sigma(r)}_{i_\tau(r)} .$$

Clearly $D^{(r)}$ is self-adjoint under the involution $\dagger$ and its representation in the form (2.6) is just

$$D^{(r)} = \sum_{1 \leq i_1 < \cdots < i_r \leq N} \sum_{\sigma, \tau \in \mathscr{S}_r} \operatorname{sign}(\sigma\tau)\, \Lambda^{i_\sigma(1)}_{i_\tau(1)} \otimes \cdots \otimes \Lambda^{i_\sigma(r)}_{i_\tau(r)} + \mathscr{J} .$$

## 3. Quantum stochastic calculus

We are concerned with the quantum stochastic calculus in the Fock space $\mathscr{F}(\mathfrak{h})$ over the Hilbert space

$$\mathfrak{h} = L^2(\mathbb{R}_+, \mathbb{C}^N) = L^2(\mathbb{R}_+) \otimes \mathbb{C}^N \tag{3.1}$$

of vector-valued functions on $\mathbb{R}_+$ taking values in $\mathbb{C}_N$, and with the class of integrator processes $\big(\Lambda_H(t),\, t \in \mathbb{R}_+\big)$ labelled by $H \in \mathscr{L}$, which are conveniently defined by

$$\big\langle \Psi(f),\, \Lambda_H(t)\Psi(g) \big\rangle = \int_0^t \big\langle f(s),\, Hg(s) \big\rangle\, ds \big\langle \Psi(f),\, \Psi(f) \big\rangle , \tag{3.2}$$

where $\Psi(f)$ is the exponential vector in $\mathscr{F}(\mathfrak{h})$ corresponding to $f \in \mathfrak{h}$. Then we may define in particular iterated stochastic integral processes of the form

$$I^n_t(H_1, \ldots, H_n) = \int_{0<t_1<\cdots<t_n<t} d\Lambda_{H_1}(t_1) \ldots d\Lambda_{H_n}(t_n), \quad H_1 \ldots, H_n \in \mathscr{L} ,$$

for which

$$\big\langle \Psi(f),\, I^n_t(H_1, \ldots, H_n)\Psi(g) \big\rangle =$$
$$= \int_{0<t_1<\cdots<t_n<t} \big\langle f(t_1), H_1 g(t_1) \big\rangle \ldots \big\langle f(t_n),\, H_n g(t_n) \big\rangle\, dt_1 \ldots dt_n \big\langle \Psi(f), \Psi(g) \big\rangle . \tag{3.3}$$

Note that

$$I^n_t(H_1, \ldots, H_n)^\dagger = I^n_t(H_1^\dagger, \ldots, H_n^\dagger) , \tag{3.4}$$

where, on the lhs, † denotes the restriction to the exponential domain $\mathscr{E}$ of the Hilbert space adjoint, and that

$$\Lambda_H(t) = I_t^1(H). \tag{3.5}$$

It is evident from (3.3) that $I_t^n(H_1, \ldots, H_n)$ is multilinear in $H_1, \ldots, H_n$, hence $I_t^n$ extends uniquely to a linear map from the $n$-fold tensor product $\mathscr{L}\otimes\cdots\otimes\mathscr{L}$. We can thus define an iterated integral process $I(T) = \{I_t(T),\, t \in \mathbb{R}_+\}$ for each element $T$ of the tensor algebra $\mathscr{T}$ by linear extension of the maps $I^n$ defined on the homogeneous subspaces of $\mathscr{T}$. Using the independence of the stochastic integrators $\Lambda_H$ it can be seen that $I_t(T) = 0$ for $t > 0$ only if $T = 0$.

Unfortunately $I$ does not vanish on the ideal $\mathscr{J}$ so we cannot define $I(X)$ unambiguously for $X \in \mathscr{U}$.

## 4. A product formula for iterated integrals

In what follows operators $R$ and $S$ on the exponential domain $\mathscr{E}$ are said to have product $T = RS$ if, for all $f, g \in h$,

$$\langle \Psi(f),\, T\Psi(g)\rangle = \langle R^\dagger \Psi(f),\, S\Psi(g)\rangle.$$

In fact, for the products which concern us, formed from the operators $\Lambda_H(t)$, this definition is equivalent to the usual operator product if the exponential domain is suitably enlarged [5].

**Proposition 4.1.** *Let $H, K_1, \ldots, K_m$ be operators on $\mathbb{C}^N$. Then*

$$I_t^1(H) I_t^m(K_1, \ldots, K_m) = \sum_{j=0}^{m} I_t^{m+1}(K_1, \ldots, K_j,\, H,\, K_{j+1}, \ldots, K_m) + \\ + \sum_{j=1}^{m} I_t^m(K_1, \ldots, HK_j,\, K_{j+1}, \ldots, K_m).$$

P r o o f. The case $m = 1$, that

$$I_t^1(H) I_t^1(K) = I_t^2(H, K) + I_t^2(K, H) + I_t^1(HK), \tag{4.1}$$

follows from the quantum Ito formula of [2, 5] using (3.6). Assuming that the case $m = n - 1$ holds, we have, by the same formula

$$d\{I_t^1(H)\, I_t^n(K_1, \ldots, K_n)\} = dI_t^{n+1}(K_1, \ldots, K_n,\, H) + \\ + I_t^1(H) I_t^{n-1}(K_1, \ldots, K_{n-1}) d\Lambda_{K_n}(t) = \\ = I_t^{n-1}(K_1, \ldots, K_{n-1}) d\Lambda_{HK_n}(t).$$

Applying the case $m = n - 1$ to the middle term and integrating yields the result. □

**COROLLARY 4.2.**

$$I^t_m(K_1,\dots,K_m)I^t_1(H)=\sum_{j=0}^{m}I^t_{m+1}(K_1,\dots,K_j,\,H,\,K_{j+1},\dots,K_m)+$$
$$+\sum_{j=1}^{m}I^t_m(K_1,\dots,K_{j-1},\,K_jH,\,K_{j+1},\dots,K_m)\,. \tag{4.2}$$

Proof. Clear by taking adjoints and using (3.4). □

**PROPOSITION 4.3.** *Let $H_1,\dots,H_m,\ H_{m+1},\dots,H_{m+n}$ be operators on $\mathbb{C}^N$. Then*

$$I^m_t(H_1,\dots,H_m)I^n_t(H_{m+1},\dots,H_{m+n})=$$
$$=\sum_{k=\max\{m,n\}}^{m+n}\ \sum_{(S_1,\dots,S_k)\in\mathbb{P}_k}I^k_t(H_{S_1},\dots,H_{S_k}) \tag{4.3}$$

*where $\mathbb{P}_k$ is the set of ordered partitions $(S_1,\dots,S_k)$ of $\{1,\dots,m+n\}$ such that*

(a) *each $S$ is either a singleton $\{i\}$, or a pair $\{j,l\}$ with $j\in\{1,\dots,m\}$ and $l\in\{m+1,\dots,m+n\}$. In the former case $H_S=H_i$, in the latter $H_S=H_jH_l$,*

(b) *in the permutation $(S_1,\dots,S_k)$ of $\{1,\dots,m+n\}$ in which each $S$ is written in increasing order, the numbers $1,\dots,m$ occur in their natural order, as do the numbers $m+1,\dots,m+n$.*

*Thus, for example*

$$I^2_t(H,K)I^2_t(L,M)=$$
$$=I^4_t(H,L,M,K)+I^4_t(L,H,M,K)+I^4_t(L,M,H,K)+$$
$$+I^4_t(L,H,K,M)+I^4_t(H,L,K,M)+I^4_t(H,K,L,M)+$$
$$+I^3_t(HL,M,K)+I^3_t(L,HM,K)+I^3_t(H,KL,M)+$$
$$+I^3_t(HL,K,M)+I^3_t(H,L,KM)+I^3_t(L,H,KM)+$$
$$+I^2_t(HL,KM)\,.$$

Proof. Proposition 4.1 and Corollary 4.2 establish the cases when $m=1$ and $n=1$ respectively. By the quantum Ito formula we have

$$d\big(I_m(H_1,\dots,H_m)\,I_n(H_{m+1},\dots,H_{m+n})\big)=$$
$$=I_{m-1}(H_1,\dots,H_{m-1})\,I_n(H_{m+1},\dots,H_{m+n})d\Lambda_{H_m}+$$
$$+I_m(H_1,\dots,H_m)\,I_{n-1}(H_{m+1},\dots,H_{m+n-1})d\Lambda_{H_{m+n}}+$$
$$+I_{m-1}(H_1,\dots,H_{m-1})\,I_{n-1}(H_{m+1},\dots,H_{m+n-1})d\Lambda_{H_mH_{m+n}}\,. \tag{4.4}$$

Making the inductive assumption that the Proposition holds when $(m,n)$ is replaced by any of $(m-1,n)$, $(m,n-1)$ or $(m-1,n-1)$ and integrating, we obtain the result, noting that the three terms on the right hand side of (4.4) account for the terms on the right hand side of (4.3) in which $H_{S_k}$ takes the three possible values $H_m$, $H_{m+n}$ and $H_m\, H_{m+n}$ respectively. □

Writing (4.3) in the form

$$I(H_1 \otimes \cdots \otimes H_m)\, I(H_{m+1} \otimes \cdots \otimes H_{m+n}) =$$

$$= \sum_{k=\max\{m,n\}}^{m+n} \ \sum_{(S_1,\dots,S_k)\in\mathbb{P}_k} I(H_{S_1} \otimes \cdots \otimes H_{S_k})$$

and using the faithfulness of the iterated integral map $I$, we see that there is a bilinear composition $*$ in the tensor algebra $\mathscr{I}$ such that

$$I(X)\, I(Y) = I(X * Y) \quad (X, Y \in \mathscr{I}).$$

It may be verified directly from (4.3) that this composition is associative; alternatively of course this is clear from the equivalence to the usual multiplication of operators noted at the beginning of this section. In general, for Casimir elements

$$C = c + \mathscr{J}, \quad D = d + \mathscr{J}$$

with $c$ and $d$ symmetric, $c * d$ is not symmetric. However, as the next proposition shows, $c * d + \mathscr{J}$ is a Casimir element and thus defines an associative bilinear composition $\circ$ in $\mathscr{Z}$ by

$$C \circ D = c * d + \mathscr{J},$$

which we call the *chaotic product.*

**PROPOSITION 4.4.** *Suppose that elements $c, d \in \mathscr{I}$ are pointwise invariant under the adjoint action of $U(N)$. Then so too is $c * d$.*

Proof. For $U \in U(N)$ and $t \in \mathbb{R}_+$ let $\pi_t(U)$ be the second quantisation unitary operator on $\mathscr{F}(\mathfrak{h})$ whose action on exponential vectors is

$$\pi_t(U)\Psi(f) = \Psi(\chi_{[0,t)} U f + \chi_{(t,\infty)} f);$$

$\pi_t$ is the unitary representation of $U(N)$ generated by the representation $H \mapsto \Lambda_H(t)$ of $\mathscr{L}$. Then it is easily seen from (3.3) that, for all $X \in \mathscr{I}$,

$$\pi_t(U)\, I_t(X)\, \pi_t(U)^{-1} = I_t(\mathrm{ad}_U X).$$

It follows that

$$\begin{aligned} I_t(\mathrm{ad}_U\, c * d) &= \pi_t(U)\, I_t(c * d)\, \pi_t(U)^{-1} \\ &= \pi_t(U)\, I_t(c)\, I_t(d) \pi_t(U)^{-1} \\ &= \pi_t(U)\, I_t(c)\, \pi_t(U)^{-1} \pi_t(U)\, I_t(d)\, \pi_t(U)^{-1} \\ &= I_t(\mathrm{ad}_U\, c)\, I_t(\mathrm{ad}_U\, d) \\ &= I_t(c)\, I_t(d) \\ &= I_t(c * d). \end{aligned}$$

□

## 5. Casimir processes and Casimir chaos processes

Consider a Casimir element $C$, represented in the form

$$C = c + \mathscr{J},$$

with $c$ a symmetric tensor. The family of infinitesimal representations $\delta\pi_t \colon H \mapsto \Lambda_H(t)$ of the unitary representations $\pi_t$ of $U(N)$ extend naturally to the universal enveloping algebra $\mathscr{U}$, and thus generates a *Casimir process* $(C_t,\, t \geq 0)$ given by

$$C_t = \delta\pi_t(C).$$

It is clear that, for $C, D \in \mathscr{Z}$

$$(CD)_t = C_t D_t, \tag{5.1}$$

where on the left the multiplication is the usual one in $\mathscr{U}$, and on the right the weak operator product introduced in § 4.

**Proposition 5.1.** *Each Casimir process* $(C_t,\, t \in \mathbb{R}_+)$ *is commutative;* $C_s C_t = C_t C_s$ *for* $s \neq t$.

P r o o f . We adapt the argument of [3]. Assume without loss of generality that $s < t$. Then [2, 5] each $\Lambda_H(t)$, $H \in \mathscr{L}$ can be expressed in the form

$$\Lambda_H(t) = \Lambda_H(s) + \Lambda^{s,t}(H),$$

where $\Lambda^{s,t}(H)$ commutes with all $\Lambda_K(s)$, and hence with $C_s$. By the centrality of $C$, $C_s$ also commutes with $\Lambda_H(s)$, hence it commutes with $\Lambda_H(t)$. But then it must commute with any polynomial in the $\Lambda_H(t)$, $H \in \mathscr{L}$, in particular with $C_t$. □

N o t e s .

1. The same argument shows that the set of all Casimir processes is commutative.
2. If $C = C^\dagger$ then (because $U(N)$ is compact) the operators $C_t$ are essentially self-adjoint and thus give rise to a classical stochastic process in any state. For example in a coherent state $\hat{\Psi}(f) = \|\Psi(f)\|^{-1}\Psi(f)$, $(D_t^{(1)},\, t \geq 0)$ is a Poisson process of intensity measure $\|f(t)\|^2 dt$.

We may also associate with each $C$ in $\mathscr{Z}$ a second adapted process $(\tilde{C}_t,\, t \in \mathbb{R}_+)$ called the *Casimir chaos process.* This is defined by

$$\tilde{C}_t = I_t(c).$$

From § 4 we know that

$$(\tilde{C} \circ \tilde{D})_t = \tilde{C}_t \tilde{D}_t.$$

In [3] we showed that, in the case of the "determinantal" Casimir elements $D^{(r)}$ of § 2,

$$D_t^{(r)} = \sum_{s=1}^{r} \alpha_{rs} \tilde{D}_t^{(s)}, \tag{5.3}$$

where the coefficients $\alpha_{rs}$ are all non-zero and do not depend on $t$. In particular this shows that in general $C_t \neq \tilde{C}_t$.

**PROPOSITION 5.2.** *The injective maps* $\Phi\colon C \to (C_t)$, $\tilde{\Phi}\colon C \to (\tilde{C}_t)$ *have the same range.*

Proof. The elements $D^{(r)}$, $r = 1, \dots, N$, generate $\mathscr{X}$, hence from (5.1) their images $(D_t^{(n)}, t \geq 0)$ generate the range of $\Phi$ under operator multiplication. The result follows from the invertibility of the triangular linear relation (5.3). □

**COROLLARY 5.2.** *There exists a bijective linear map* $\sim\colon \mathscr{X} \to \mathscr{X}$, $C \mapsto \tilde{C}$ *such that*

$$(a)\tilde{\Phi}(C) = \Phi(\tilde{C}) \quad \text{for all} \quad C \in \mathscr{X},$$
$$(b)(C \circ D)^{-} = \tilde{C}\tilde{D}.$$

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