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STATES AND OBSERVABLES ON MV ALGEBRAS

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ABSTRACT. In this paper, states and observables on MV algebras, as special cases of a D-morphism of MV algebras, are studied.

1. Introduction

The axiomatic Kolmogorov model of probability theory is formed by the triplet $(\Omega, \mathcal{S}, P)$, where Ω is a non-empty set, $\mathcal{S}$ is a σ-algebra of subsets of Ω, and P is a probability measure. Another fundamental notion of probability theory is a random variable. A random variable is a pointwise function $\xi : \Omega \to \mathbb{R}$ such that $\xi^{-1}(E) \in \mathcal{S}$ for every Borel set E. Then ξ^{-1} is a σ-homomorphism from the σ-algebra $\mathcal{B}(\mathbb{R})$ of Borel sets of the real line $\mathbb{R}$ into the σ-algebra $\mathcal{S}$. A mapping $P_\xi : \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by formula $P_\xi(E) = P(\xi^{-1}(E))$ for every $E \in \mathcal{B}(\mathbb{R})$ is a probability on $\mathcal{B}(\mathbb{R})$ (a probability distribution of a random variable ξ).

States and observables are two basic notions of quantum logics theory. A state plays the same role as a probability measure in classical probability theory and an observable is an analogue of a random variable.

C. C. Chang in [1] developed theory of algebraic systems that correspond in a natural way to the $\aleph_0$-valued propositional calculus. These algebraic systems are called MV algebras, where MV is supposed to suggest many-valued logics. The classical two-valued logic gives a rise to the study of Boolean algebras and, every Boolean algebra is an MV algebra where as the converse does not hold.

Recently, a very simple but very general structure has appeared, so called a D-poset [4], which was inspired by an investigation of the possibility to introduce fuzzy set ideas to quantum structures models [3]. D-posets are a natural generalization of, for example, quantum logics, real vector lattices, orthoalgebras and MV algebras. The states and observables on D-posets are defined as the special cases of morphisms of D-posets.

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For the first time the notions of a state, an observable and a joint observable on an MV algebra of fuzzy sets have been defined by B. Riečan in [6].

In the present paper, we introduce the notions of a state and an observable on a general MV algebra in a similar way as in D-posets.

2. MV algebras

In [5] an MV algebra is defined as follows:
An *MV algebra* is an algebra $(A, \oplus, \odot, *, 0, 1)$, where A is a non-empty set, 0 and 1 are constant elements of A, $\oplus$ and $\odot$ are binary operations, and $*$ is a unary operation, satisfying the following axioms:

(2.1) $(a \oplus b) = (b \oplus a)$;
(2.2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(2.3) $a \oplus 0 = a$;
(2.4) $a \oplus 1 = 1$;
(2.5) $(a^*)^* = a$;
(2.6) $0^* = 1$;
(2.7) $a \oplus a^* = 1$;
(2.8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$;
(2.9) $a \odot b = (a^* \oplus b^*)^*$.

The lattice operations $\vee$ and $\wedge$ are defined by the formulas

$$a \vee b = (a \odot b^*) \oplus b \quad \text{and} \quad a \wedge b = (a \oplus b^*) \odot b .$$

We write $a \le b$ iff $a \vee b = b$. The relation $\le$ is a partial ordering over A and $0 \le a \le 1$, for every $a \in A$. An MV algebra is a distributive lattice with respect to the operations $\vee$, $\wedge$.

In [1] the following assertions have been proved:

(2.10) $a \odot b \le a \wedge b \le a \vee b \le a \oplus b$, for every $a, b \in A$.
(2.11) If $a \le b$, then $a \oplus c \le b \oplus c$ and $a \odot c \le b \odot c$, for every $c \in A$.
(2.12) The following three conditions are equivalent:
(i) $a \le b$, (ii) $a^* \oplus b = 1$, (iii) $a \odot b^* = 0$.
(2.13) If $a \le b$, then $b = a \oplus (b \odot a^*)$.
(2.14) $(a \vee b)^* = a^* \wedge b^*$ and $(a \wedge b)^* = a^* \vee b^*$.

Example 2.1. Every Boolean algebra is an MV algebra. Especially, let B be an algebra of subsets of a non-empty set X. We put $E \oplus F = E \cup F$, $E \odot F = E \cap F$, $E^* = X \setminus E$, for every $E, F \in B$, $0 = \emptyset$ and $1 = X$. Then B forms an MV algebra, where $\le$ is the inclusion relation.

Example 2.2. Let $\mathcal{I}$ be a subset of the interval [0,1] of real numbers such that $0 \in \mathcal{I}$, $1 \in \mathcal{I}$, and if $a, b \in \mathcal{I}$, then $a \oplus b := \min(1, a+b) \in \mathcal{I}$, $a \odot b := \max(0, a+b-1) \in \mathcal{I}$, $a^* := 1 - a \in \mathcal{I}$, where $+$ and $-$ denote the usual sum and difference of real numbers. The system $\mathcal{I}$ is an MV algebra. Moreover, $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$ and the relation $\leq$ is the natural ordering of real numbers. It is not difficult to show that, for $a, b \in \mathcal{I}$, $a \oplus b = a + b$ if and only if $a \leq b^* = 1 - b$ and $a \leq b$ implies $b \odot a^* = b - a$.

Example 2.3. Let X be a non-empty set and $\mathcal{F} \subseteq [0,1]^X$ be a system of functions $f\colon X \to [0,1]$ such that:

(i) if $0(t) = 0$ and $1(t) = 1$ for every $t \in X$, then $0 \in \mathcal{F}$ and $1 \in \mathcal{F}$;
(ii) if $f, g \in \mathcal{F}$, then $f \oplus g := \min(1, f+g) \in \mathcal{F}$ and $f \odot g := \max(0, f+g-1) \in \mathcal{F}$, where $(f+g)(t) = f(t) + g(t)$, for every $t \in X$;
(iii) if $f \in \mathcal{F}$, then $f^* := (1 - f) \in \mathcal{F}$.

Then the system $\mathcal{F}$ is an MV algebra (of fuzzy sets). We point out here that the operations $\vee$ and $\wedge$ on $\mathcal{F}$ are $f \vee g = \max(f,g)$, $f \wedge g = \min(f,g)$, and $f \leq g$ iff $f(t) \leq g(t)$, for every $t \in X$.

Let $\mathcal{A}$ be an MV algebra. We define a binary operation $\setminus$ on $\mathcal{A}$ (a difference) by the formula

$$b \setminus a := b \odot a^* \quad \text{for any} \quad a, b \in \mathcal{A}.$$

The difference operation on an MV algebra is already known in the literature and it has been studied in [2], for example.
It is evident that $1 \setminus a = a^*$, $a \setminus 0 = a$, $a \setminus a = 0$ and $b \setminus a \leq b$, for any $a, b \in \mathcal{A}$.

Proposition 2.4. *Let $\mathcal{A}$ be an MV algebra and $a, b, c \in \mathcal{A}$. Then:*

(i) $a \leq b$ *implies* $b \setminus (b \setminus a) = a$, *and* $b = a \oplus (b \setminus a)$;
(ii) $a \leq b^*$ *implies* $a = (a \oplus b) \setminus b$.
(iii) $a \leq b \leq c$ *implies*
$c \setminus b \leq c \setminus a$ *and* $(c \setminus a) \setminus (c \setminus b) = b \setminus a$,
$b \setminus a \leq c \setminus a$ *and* $(c \setminus a) \setminus (b \setminus a) = c \setminus b$.

Proof. The assertions (i) and (ii) follow from the definitions of the lattice operations $\wedge$ and $\vee$. Indeed,

$$b \setminus (b \setminus a) = b \odot (b \odot a^*)^* = b \odot (b^* \oplus a) = a \wedge b,$$
$$a \oplus (b \setminus a) = a \oplus (b \odot a^*) = a \vee b, \text{ and}$$
$$(a \oplus b) \setminus b = (a \oplus b) \odot b^* = a \wedge b^*.$$

(iii) If $a \leq b \leq c$, then $c^* \leq b^* \leq a^*$ and, therefore, $c \setminus b = c \odot b^* \leq c \odot a^* = c \setminus a$. Calculate

$(c \setminus a) \setminus (c \setminus b) = (c \odot a^*) \odot (c \odot b^*)^* = a^* \odot \big(c \odot (c^* \oplus b)\big) =$
$= a^* \odot (b \wedge c) = a^* \odot b = b \setminus a\,.$
Similarly,
$(b \setminus a) = b \odot a^* \le c \odot a^* = c \setminus a$, and $(c \setminus a) \setminus (b \setminus a) = (c \odot a^*) \odot (b \odot a^*)^* =$
$= c \odot \big(a^* \odot (b^* \oplus a)\big) = c \odot (a^* \wedge b^*) = c \odot b^* = c \setminus b\,.$

□

DEFINITION 2.5. [4] Let $\mathcal{P}$ be a partially ordered set with a partial ordering $\le$, the greatest element 1, and with a partial binary operation $\setminus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, called a difference, such that, for $a, b \in \mathcal{P}$, $b \setminus a$ is defined iff $a \le b$, and the following three axioms hold for $a, b, c \in \mathcal{P}$:

(i) $b \setminus a \le b$;
(ii) $b \setminus (b \setminus a) = a$;
(iii) $a \le b \le c$ implies $c \setminus b \le c \setminus a$ and $(c \setminus a) \setminus (c \setminus b) = b \setminus a$.

Then $\mathcal{P}$ is called a *D-poset* or a *difference poset.*

From the introduced above it follows that any MV algebra is a D-poset, in particular, any MV algebra of fuzzy sets (see Example 2.3) is a D-poset of fuzzy sets [3].

PROPOSITION 2.6. *Let $\mathcal{A}$ be an MV algebra. Then:*

(i) $(a \vee b) \setminus a = b \setminus (a \wedge b)$;
(ii) $(a \oplus b) \setminus a = b \setminus (a \odot b)$;
(iii) $(a \oplus b) \setminus (a \vee b) = (a \wedge b) \setminus (a \odot b)$.

Proof. Using (i) and (ii) from Proposition 2.4 we obtain:

(i) $(a \vee b) \setminus a = \big(a \oplus (b \setminus a)\big) = b \setminus a = b \setminus \big(b \setminus (b \setminus a)\big) = b \setminus (a \wedge b)$.
(ii) $(a \oplus b) \setminus a = (a \oplus b) \odot a^* = (a^* \oplus b^*) \odot b = (a \odot b)^* \odot b = b \setminus (a \odot b)$.
(iii) $(a \oplus b) \setminus (a \vee b) = (a \oplus b) \odot \big(a \oplus (a^* \odot b)\big)^* = (a \oplus b) \odot \big(a^* \odot (a \oplus b^*)\big) =$
$= \big((a \oplus b) \odot a^*\big) \odot (a \oplus b^*) = \big((a^* \oplus b^*) \odot b\big) \odot (a \oplus b^*) = \big((a^* \oplus b^*) \odot$
$(b \odot (a \oplus b^*))\big) = (a \odot b)^* \odot (a \wedge b) = (a \wedge b) \setminus (a \odot b)$.

□

PROPOSITION 2.7. *Let $a \le b$ and $d \le c$. If $b \setminus a = c \setminus d$, then $a \oplus c = b \oplus d$.*

Proof. By (i) of Proposition 2.4 we have $b = a \oplus (b \setminus a)$ and $c = d \oplus (c \setminus d)$. Then $a \oplus c = a \oplus \big(d \oplus (c \setminus d)\big) = a \oplus \big(d \oplus (b \setminus a)\big) = \big(a \oplus (b \setminus a)\big) \oplus d = b \oplus d$. □

The converse assertion, in general, is not true. Indeed, let $\mathcal{I}$ be an MV algebra from Example 2.2. Put $a = 0.2$, $b = 0.7$, $c = 0.9$ and $d = 0.3$. Then $b \oplus d = 1 = a \oplus c$, but $b \setminus a = 0.5$ and $c \setminus d = 0.6$.

COROLLARY 2.8.

(i) $(a \vee b) \oplus (a \wedge b) = a \oplus b = (a \oplus b) \oplus (a \odot b)$;
(ii) *if* $a \le b \le c$, *then* $(c \setminus b) \oplus (b \setminus a) = c \setminus a$;
(iii) $(a \wedge b) \oplus (b \setminus a) = b$.

PROPOSITION 2.9. *Let* $\mathcal{A}$ *be an MV algebra. Then:*

(i) $c \setminus (a \wedge b) = (c \setminus a) \vee (c \setminus b)$;
(ii) $c \setminus (a \vee b) = (c \setminus a) \wedge (c \setminus b)$;
for any $a, b, c \in \mathcal{A}$.

Proof.

(i) By [1; Ax. 11'] $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ for all x, y, z from an MV algebra $\mathcal{A}$. It suffices to put $x = c$, $y = a^*$ and $z = b^*$. Then $c \setminus (a \wedge b) = c \odot (a \wedge b)^* = c \odot (a^* \vee b^*) = (c \odot a^*) \vee (c \odot b^*) = (c \setminus a) \vee (c \setminus b)$.

(ii) Since $a \le a \vee b$ and $b \le a \vee b$, it is evident that $c \setminus (a \vee b) \le c \setminus a$ and $c \setminus (a \vee b) \le c \setminus b$, which gives that $c \setminus (a \vee b)$ is a lower bound of the set $\{c \setminus a, c \setminus b\}$. If $d \in \mathcal{A}$, $d \le c \setminus a$ and $d \le c \setminus b$, then $c \wedge a = c \setminus (c \setminus a) \le c \setminus d$ and $c \wedge b = c \setminus (c \setminus b) \le c \setminus d$.
Hence $(c \wedge a) \vee (c \wedge b) = c \wedge (a \vee b) \le c \setminus d$, therefore, $d = c \wedge d = c \setminus (c \setminus d) \le c \setminus \big(c \wedge (a \vee b)\big) = (c \setminus c) \vee \big(c \setminus (a \vee b)\big) = c \setminus (a \vee b)$.
Thus $c \setminus (a \vee b)$ is the greatest lower bound of $\{c \setminus a, c \setminus b\}$, i.e. $c \setminus (a \vee b) = (c \setminus a) \wedge (c \setminus b)$. □

3. States and observables on MV algebras

According to [1], we will say that a mapping w from an MV algebra $\mathcal{A}$ into an MV algebra $\mathcal{B}$ is a *homomorphism* of MV algebras or an *MV-homomorphism* iff $w(0_{\mathcal{A}}) = 0_{\mathcal{B}}$, $w(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, and w preserves the operations $\oplus, \odot$ and $*$.

PROPOSITION 3.1. *A mapping* $w \colon \mathcal{A} \to \mathcal{B}$ *is an MV-homomorphism if and only if* $w(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, *and* w *preserves difference operations on* $\mathcal{A}$ *and* $\mathcal{B}$.

Proof. The necessary condition is evident.
Let $w(b \setminus_{\mathcal{A}} a) = w(b) \setminus_{\mathcal{B}} w(a)$ for any $a, b \in \mathcal{A}$, where $\setminus_{\mathcal{A}}$ and $\setminus_{\mathcal{B}}$ denote difference operations on $\mathcal{A}$ and $\mathcal{B}$, respectively. We will write shortly $\setminus$ for both operations. Then:
(i) $w(0_{\mathcal{A}}) = w(0_{\mathcal{A}} \setminus 0_{\mathcal{A}}) = w(0_{\mathcal{A}}) \setminus w(0_{\mathcal{A}}) = w(0_{\mathcal{A}}) \odot (w(0_{\mathcal{A}})^* = 0_{\mathcal{B}}$.
(ii) $w(a^*) = w(1_{\mathcal{A}} \setminus a) = w(1_{\mathcal{A}}) \setminus w(a) = 1_{\mathcal{B}}) \setminus w(a) = w(a)^*$.
(iii) $w(a \odot b) = w(a \setminus b^*) = w(a) \setminus w(b)^* = w(a) \odot w(b)$.
(iv) $w(a \oplus b) = w\big((a^* \odot b^*)^*\big) = \big(w(a^*) \odot w(b^*)\big)^* = w(a) \oplus w(b)$. □

Let $w\colon \mathcal{A} \to \mathcal{B}$ be an MV-homomorphism. The *range* of w is the set $\mathcal{R}(w) = \{w(a)\colon a \in \mathcal{A}\}$. Then $\mathcal{R}(w)$ is an MV algebra of $\mathcal{B}$ (see [1; p. 471]). It is easy to see that if $\mathcal{A}$ is a Boolean algebra, then $\mathcal{R}(w)$ is a Boolean subalgebra of $\mathcal{B}$. If we put, in this case, $\mathcal{B} = \mathcal{I}$, where $\mathcal{I}$ is the MV algebra of all reals from the interval $[0, 1]$, then $\mathcal{R}(w) = \{0, 1\}$.

DEFINITION 3.2. An MV algebra $\mathcal{A}$ is said to be an MV σ-algebra, if each countable sequence of elements from $\mathcal{A}$ has the supremum in $\mathcal{A}$.

It is clear that every Boolean σ-algebra is an MV σ-algebra. The MV algebra $\mathcal{I}$ from Example 2.2 and the MV algebra $\mathcal{F}$ from Example 2.3 are MV σ-algebras, too.

DEFINITION 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two MV algebras (MV σ-algebras). A mapping $w\colon \mathcal{A} \to \mathcal{B}$ is called an *D-morphism* (an *D-σ-morphism*) if the following conditions are satisfied:

(3.1) $w(1_{\mathcal{A}}) = 1_{\mathcal{B}}$;

(3.2) if $a, b \in \mathcal{A}$, $a \le b$, then $w(a) \le w(b)$ and $w(b \setminus a) = w(b) \setminus w(a)$;

(3.3) if $(a_n)_{n=1}^{\infty} \subseteq \mathcal{A}$, $a_n \nearrow a$ (i.e., $a_n \le a_{n+1}$ for any $n \in \mathbb{N}$ and $a = \bigvee\limits_{n=1}^{\infty} a_n$), then $w(a_n) \nearrow w(a)$.

PROPOSITION 3.4. *Let $w\colon \mathcal{A} \to \mathcal{B}$ be a D-morphism of MV algebras $\mathcal{A}$ and $\mathcal{B}$. Then the following assertions are true.*

(i) $w(0_{\mathcal{A}}) = 0_{\mathcal{B}}$.

(ii) $w(a^*) = w(a)^*$.

(iii) $w(a \vee b) = w(a) \oplus w(b \setminus a) = w(b) \oplus w(a \setminus b)$.

(iv) *If $a \le b$, then $w(b) = w(a) \oplus \big((w(b) \setminus w(a)\big)$.*

(v) $w(a \oplus b) \oplus w(a \odot b) = w(a) \oplus w(b) = w(a \vee b) \oplus w(a \wedge b)$.

(vi) *If $a \le b^*$, then $w(a \oplus b) = w(a) \oplus w(b)$.*

Proof.

(i) $w(0_{\mathcal{A}}) = w(a \setminus a) = w(a) \setminus w(a) = 0_{\mathcal{B}}$.

(ii) $w(a^*) = w(1_{\mathcal{A}} \setminus a) = w(1_{\mathcal{A}}) \setminus w(a) = w(a)^*$.

(iii) By the monotonicity of the D-morphism w, we have $w(a) \le w(a \vee b)$ and, using (2.4), (3.2) and (i) of Proposition 2.4, we get

$w(a \vee b) = w(a) \oplus \big(w(a \vee b) \odot w(a)^*\big) = w(a) \oplus w\big((a \vee b) \setminus a\big) = w(a) \oplus w(b \setminus a)$

and dually $w(a \vee b) = w(b) \oplus w(a \setminus b)$.

(iv) This result follows directly from (iii) and (3.2).

(v) By the inequalities $a \odot b \leq b, \quad a \leq a \oplus b$ and by (iv) we have

$w(b) = w(a \odot b) \oplus \big(w(b) \setminus w(a \odot b)\big)$ and

$w(a \oplus b) = w(a) \oplus \big(w(a \oplus b) \setminus w(a)\big) = w(a) \oplus w\big((a \oplus b) \setminus a\big) = = w(a) \oplus \big(w(b) \setminus w(a \odot b)\big)$.

Now we calculate,

$w(a) \oplus w(b) = w(a) \oplus w(a \odot b) \oplus \big(w(b) \setminus w(a \odot b)\big) = w(a \odot b) \oplus w(a \oplus b)$.

The equality $w(a \oplus b) \oplus w(a \odot b) = w(a \vee b) \oplus w(a \wedge b)$ follows from (iii) of Proposition 2.4 and from Proposition 2.6.

(vi) If $a \leq b^*$, then $a \odot b = 0_{\mathcal{A}}$ and, using (v), we obtain that

$w(a) \oplus w(b) = w(a \odot b) \oplus w(a \oplus b) = 0_{\mathcal{B}} \oplus w(a \oplus b) = w(a \oplus b)$.

□

The following assertions follow from (v) of Proposition 3.4.

COROLLARY 3.5.

(i) $w(a) = 0_{\mathcal{B}}$ *implies* $w(a \oplus b) = w(b) = w(a \vee b)$ *for any* $b \in \mathcal{A}$;
(ii) $w(a) = 1_{\mathcal{B}}$ *implies* $w(a \odot b) = w(b) = w(a \wedge b)$ *for any* $b \in \mathcal{A}$.

DEFINITION 3.6. Let $\mathcal{A}$ be an MV σ-algebra and $\mathcal{I}$ be an MV σ-algebra of reals from the interval [0,1]. A D-σ-morphism $s\colon \mathcal{A} \to \mathcal{I}$ is said to be a *state* (on $\mathcal{A}$).

Let $\mathcal{B}(\mathbb{R})$ be the MV σ-algebra of all Borel subsets of the real line $\mathbb{R}$. A D-σ-morphism $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ is said to be an *observable* (on $\mathcal{A}$).

It is easy to see that if s is a state on an MV σ-algebra $\mathcal{A}$, then:

(3.4) $s(a^*) = 1 - s(a)$;
(3.5) $a \leq b^*$ implies $s(a \oplus b) = s(a) + s(b)$;
(3.6) if $(a_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$, $a_n \nearrow a$, then $s(a) = \lim\limits_{n \to \infty} s(a_n)$ and

$$s(a) = s(a_1) + \sum_{n=2}^{\infty} s(a_n \setminus a_{n-1});$$

(3.7) $a \leq a^*$ implies $s(a \oplus a) = 2s(a)$;
(3.8) $a^* \leq a$ implies $s(a \odot a) = 2s(a) - 1$.

EXAMPLE 3.7. Let $\mathcal{F} \subset [0,1]^X$ be an MV σ-algebra of fuzzy sets (see Example 2.3). Let $t_0 \in X$. A mapping $s\colon \mathcal{F} \to [0,1]$ defined by the formula $s(f) = f(t_0)$ is a state on $\mathcal{F}$.

Example 3.8. Let $\mathcal{A}$ be an MV algebra and $a \in \mathcal{A}$. A mapping $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ defined via

$$x_a(E) = \begin{cases} 1_{\mathcal{A}}, & \text{if } \{0,1\} \cap E = \{0,1\}, \\ a, & \text{if } \{0,1\} \cap E = \{1\}, \\ a^*, & \text{if } \{0,1\} \cap E = \{0\}, \\ 0_{\mathcal{A}}, & \text{if } \{0,1\} \cap E = \emptyset, \end{cases}$$

for every $E \in \mathcal{B}(\mathbb{R})$, is an observable on $\mathcal{A}$ (called an *indicator* of the element a).

We note that the range of an observable on an MV σ-algebra, in general, is not an MV algebra. Indeed, we put $\mathcal{A} = \mathcal{I}$ and $a = 0,2$. Then $\mathcal{R}(x_a) = \{1;\ 0,2;\ 0,8;\ 0\}$ and $0,2 \oplus 0,2 = 0,4 \notin \mathcal{R}(x_a)$.

Proposition 3.9. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be MV σ-algebras, $u\colon \mathcal{A} \to \mathcal{B}$ and $v\colon \mathcal{B} \to \mathcal{C}$ two MV homomorphisms (D-σ-morphisms). Then the composition $v \circ u\colon \mathcal{A} \to \mathcal{C}$ defined by the formula $v \circ u(a) = v\big(u(a)\big)$, for every $a \in \mathcal{A}$, is an MV homomorphism (a D-σ-morphism).*

The proof of this proposition requires only a routine verification of the definition of an MV homomorphism (a D-σ-morphism).

Theorem 3.10. *Let x be an observable and s be a state on an MV σ-algebra $\mathcal{A}$. Then the composition*

$s \circ x\colon \mathcal{B}(\mathbb{R}) \to [0,1]$, *where* $s \circ x(E) = s\big(x(E)\big)$, *for every* $E \in \mathcal{B}(\mathbb{R})$,
is a probability measure on $\mathcal{B}(\mathbb{R})$.

Proof. We prove only the σ-additivity of the mapping $s \circ x$. Let $\big(E_n\big)_{n=1}^{\infty}$ be a sequence of pairwise disjoint Borel subsets. Put $A_n = \bigcup_{i=1}^{n} E_i$, $n = 1, 2, \dots$.

The sequence $\big(A_n\big)_{n=1}^{\infty}$ is monotonic and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n .$$

Let us calculate

$$s\circ x\Big(\bigcup_{n=1}^{\infty} E_n\Big) = s\Big(x\Big(\bigcup_{n=1}^{\infty} E_n\Big)\Big) = s\Big(x\Big(\bigcup_{n=1}^{\infty} A_n\Big)\Big) = s\Big(\bigvee_{n=1}^{\infty} x(A_n)\Big) =$$

$$= s\big(x(A_1)\big) + \sum_{n=2}^{\infty} s\big(x(A_n)\setminus x(A_{n-1})\big) =$$

$$= s\big(x(A_1)\big) + \sum_{n=2}^{\infty} s\big(x(A_n\setminus A_{n-1})\big) =$$

$$= s\big(x(E_1)\big) + \sum_{n=2}^{\infty} s\big(x(E_n)\big) = \sum_{n=1}^{\infty} s\big(x(E_n)\big)\,.$$

□

The composition $s\circ x$ is said to be a *probability distribution* of the observable x in the state s.

Now a mean value of an observable x in a state s can be defined by the integral

$$E(x) := \int_{\mathbb{R}} t\, d(s\circ x)(t)\,,$$

if it exists and is finite.

4. Joint observables

It is well-known, from the classical probability theory, that if $(\Omega, \mathcal{S}, P)$ is a probability space, $\xi\colon \Omega\to\mathbb{R}$ and $\eta\colon \Omega\to\mathbb{R}$ are two random variables, then the random vector $T=(\xi,\eta)$ is a map from Ω into $\mathbb{R}^2$ with the property

$$T^{-1}(E\times F) = \xi^{-1}(E)\cap\eta^{-1}(F) \quad \text{for every} \quad E,F\in\mathcal{B}(\mathbb{R})\,.$$

There is an analogy between a random vector and a joint observable.

Definition 4.1. Let $\mathcal{A}$ be an MV σ-algebra and $\mathcal{B}(\mathbb{R}^2)$ be the σ-algebra of all Borel subsets of $\mathbb{R}^2$. A *joint observable* of observables x and y is a D-σ-morphism $w\colon \mathcal{B}(\mathbb{R}^2)\to\mathcal{A}$ satisfying the following identity:

$$w(E\times F) = x(E)\wedge y(F) \quad \text{for every} \quad E,F\in\mathcal{B}(\mathbb{R})\,.$$

We give a necessary condition of the existence of a joint observable.

Proposition 4.2. *If w is a joint observable of observables x and y, then*

$$x(E) \wedge y(F) = x(E) \odot y(F) \quad \textit{for every} \quad E, F \in \mathcal{B}(\mathbb{R}).$$

P r o o f. Let w be a joint observable and $E \times F \in \mathcal{B}(\mathbb{R}^2)$. Then

$$\begin{aligned} w(E \times F)^* &= w\big((E \times F)^c\big) = w(E^c \times \mathbb{R} \cup E \times F^c) = \\ &= w(E^c \times \mathbb{R}) \oplus w(E \times F^c) = x(E)^* \oplus \big(x(E) \wedge y(F)^*\big), \end{aligned}$$

therefore,

$$\begin{aligned} x(E) \wedge y(F) = w(E \times F) &= x(E) \odot \big(x(E)^* \vee y(F)\big) = \\ &= x(E) \odot \big((x(E)^* \odot y(F)^*) \oplus y(F)\big) = \\ &= x(E) \odot \big((x(E) \odot y(F)) \oplus x(E)^*\big) = \\ &= x(E) \wedge \big(x(E) \odot y(F)\big) = x(E) \odot y(F). \end{aligned}$$

□

By a joint observable we can build up the functional calculus for observables on MV σ-algebras, for example, the sum, the difference, the product, etc.

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