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ON HERGLOTZ THEOREM IN VECTOR LATTICES

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ABSTRACT. This paper is concerned with a generalization of Herglotz theorem for sequences of elements of a vector lattice.

Introduction

It is well-known that it is possible to characterize Fourier-Stieltjes coefficients of the (right-continuous) non-decreasing, bounded functions as positive definite sequences. Recall that a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is said to be positive definite if for any (complex) sequence (z_n) having only a finite number of terms different from zero we have

$$\sum_{n,m} a_{n-m} z_n \bar{z}_m \geq 0 .$$

Now according to the Herglotz theorem [1, Theorem 4.3.1] a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is positive definite if, and only if, there exists a right-continuous, non-decreasing, bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$, such that

$$a_n = \int\limits_{(-\pi,\pi]} e^{-ins} dF(s)$$

for all $n = 0, \pm 1, \dots$.

In this paper we give a generalization of the Herglotz theorem for a_n being elements of a vector lattice. As for terminology and some results from vector lattices we shall use as reference the book [2].

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Fourier-Stieltjes coefficients of vector functions of (o)-bounded variation

Recall that a function g, defined on an interval of the real line $T = [a, b]$ and taking values in a complete vector lattice Y, is said to be of (o)-bounded variation, if the set of all elements of the form

$$\sum_j |g(t_{j+1}) - g(t_j)|,$$

corresponding to all finite partitions of the interval T, is o-bounded. We shall denote by $(o)\text{-}var_{t\in T}g(t)$ the least upper bound of this set.

Denote by $BV^o(T,Y)$ the vector space of all functions on T with values in Y of o-bounded variation. Let $T = [0, 2\pi]$. Further if $g \in BV^o(T,Y)$, then an element of Y of the form

$$\hat{g}(n) = \frac{1}{2\pi} \int_T e^{-int} dg(t)$$

is called the n-th Fourier-Stieltjes coefficient of g.

In the following, let $\mathbb{T}$ denote the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ and $\mathbb{Z}$ denoting the additive group of reals and integers, respectively), as a model we may think of the interval $[0, 2\pi)$. A trigonometric polynomial on $\mathbb{T}$ is a function $a = a(t)$ defined on $\mathbb{T}$ by $a(t) = \sum_{-n}^{n} a_j e^{ijt}$. Denote by $p(\mathbb{T})$ the set of all trigonometric polynomials on $\mathbb{T}$. We shall need the following theorem [5, Theorem, 2.12] asserting that trigonometric polynomials are dense in $C(\mathbb{T})$.

THEOREM A. *For every $f \in C(\mathbb{T})$ we have $\sigma_n(f) \to f$, $n \to \infty$, in the $C(\mathbb{T})$ norm $(\|\cdot\|)$.*

We shall make use of the following result [3, Theorem 4].

THEOREM 1. *Let Y be a complete vector lattice. Let (y_k) be a two-way sequence of elements of Y. Then the following two conditions are equivalent:*

(a) *There is a function $g : T \to Y$ of (o)-bounded variation with (o)-$var_{t\in T}g(t) \le C \in Y$ such that y_j are Fourier–Stieltjes coefficients of $g(t)$, i.e.,*

$$y_j = \hat{g}(j) = \frac{1}{2\pi} \int_T e^{-ijt} dg(t) \qquad \textit{for all} \quad j \in \mathbb{Z}.$$

(b) *For all trigonometric polynomials* $a = \sum_{-l}^{l} a_j e^{ijt} \in p(T)$ *there holds*

$$\Big| \sum_{j=-l}^{l} a_{-j} y_j \Big| \le \|a\| C$$

for some $C \in Y$.

If $g \in BV^o(T, Y)$, then the (formal) series

$$\sum_{n\in\mathbb{Z}} \hat{g}(n)\, e^{inx}$$

is called the *Fourier-Stieltjes series* of g .

Let (y_j) be a two-way sequence of elements of Y . Put

$$\sigma_N(Y,t) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) y_{-j} e^{-ijt}, \quad N = 1, 2, \ldots$$

and denote by $S_N(Y)$ the (o)-bounded linear mapping on $C(\mathbf{T})$ defined by

$$S_N(Y)(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) \sigma_N(Y,t) dt, \quad f \in C(\mathbf{T}),\ N = 1, 2, \ldots$$

If the function g is of the (o)-bounded variation and $y_j = \hat{g}(j)$, $j \in \mathbb{Z}$ we shall write

$$\sigma_N(Y,t) = \sigma_N(g,t) \quad \text{and} \quad S_N(Y) = S_N(g) .$$

We have

$$S_N(Y)(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) \sigma_N(Y,t) dt$$

$$= \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j) y_{-j}, \quad f \in C(\mathbf{T}).\ N = 1, 2, \ldots .$$

Let

$$\|S_N(Y)\| = \sup_{\|f\| \le 1} \big|S_N(Y)(f)\big| .$$

We shall need also the following theorem [3, Theorem 5].

THEOREM 2. *Let Y be a complete vector lattice. The trigonometric series*

$$\sum_{n\in\mathbb{Z}} y_j e^{inx}, \quad y_j \in Y,$$

is the Fourier-Stieltjes series of the function g of the (o)-bounded variation, i.e., $y_j = \hat{g}(j)$, $j \in \mathbb{Z}$, if and only if there exists an element $0 \le C \in Y$ such that

$$\|S_N(Y)\| \le C, \quad N = 1, 2, \dots$$

It is useful to formulate the Parseval formula explicitly for the Fourier-Stieltjes series of the function g of (o)-bounded variation [3, Theorem 6].

THEOREM 3. *Let Y be a complete vector lattice and let $f \in C(T)$. Then we have*

$$\int_T f(t)dg(t) = \lim_{N\to\infty} \sum_{j=-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) \hat{f}(j)\hat{g}(-j)\,.$$

It is a very important fact that we have established not only a characterization of the Fourier-Stieltjes series of the function of (o)-bounded variation but also a method how to recapture the function by means of its Fourier-Stieltjes series. Theorem gives a recipe how to recover the function. In this sense we may, by abuse of notation, write

$$dg(t) \sim \sum_{j\in\mathbb{Z}} \hat{g}(j)e^{ijx}$$

for $g \in BV^o(T,Y)$.

It is easy to see that if the function $g\colon T \to Y$ is nondecreasing, then g is of (o)-bounded variation. Hence we may establish the following.

THEOREM 4. *Let Y be a complete vector lattice. The necessary and sufficient condition for*

$$\sum_{k\in\mathbb{Z}} y_k e^{ikx}$$

to be the Fourier-Stieltjes series of a nondecreasing function g with the values in Y is that $\sigma_N(Y,t) \ge 0$ for all N on T.

Proof. The necessity. If $y_k = \hat{g}(k)$, for a nondecreasing function g, we have

$$\sigma_N(Y,t) = \sum_{j=-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) y_{-j} e^{-ijt} = \sum_{j=-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) \hat{g}(-j) e^{-ijt} =$$

$$= \frac{1}{2\pi} \int_T \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) e^{-ij(t-s)} dg(t) = \int_T K_N(s-t)dg(t) \ge 0$$

since g is nondecreasing and Féjer's kernel K_n is nonnegative. So we have $\sigma_N(Y,t) \geq 0$ on T.

Assuming $\sigma_N(Y,t) \geq 0$ we obtain

$$\|S_N(Y)\| = \sup_{\|f\|\leq 1} \left| \int_T f(t)\sigma_N(Y,t)dt \right| = \frac{1}{2\pi} \int_T \sigma_N(Y,t)dt = y_0$$

and, by Theorem 3,

$$\sum_{j\in\mathbb{Z}} y_j e^{ijx}$$

is the Fourier-Stieltjes series for some $g \in BV^0(T,Y)$. For arbitrary nonnegative $f \in C(T)$

$$\int_T f(t)dg(t) = \lim_{N\to\infty} \frac{1}{2\pi} \int_T f(t)\sigma_N(Y,t)dt \geq 0\,,$$

hence

$$U : f \to \int_T f(t)dg(t)$$

defines a positive linear operator on $C(T)$ into Y which can be extended ([2, Theorem, 5.1.2] to the positive linear operator (denoted again by) U defined on the complete vector lattice containing characteristic functions $c_{[0,t]}$ of intervals $[0,t]$ in T. From the definition (cf. [2, Theorem, 7.1.4] $g(t) = U(c_{[0,t]})$ and it follows that g is nondecreasing.

It is not unexpected that Theorem 4 gives rise to a representation of positive-definite functions definite in a suitable sense, analogous to those known for complex-valued positive-definite functions.

Suppose that (y_n), $n = 0, \pm 1, \pm 2, \ldots$ is a two-way sequence of elements in a vector lattice Y. Then it is called positive-definite if for any sequence (c_n) of complex numbers having only a finite number of terms different from zero we have

$$\sum_{m,n} c_n \overline{c_m} y_{n-m} \geq 0\,.$$

THEOREM 5. *Let Y be a complete vector lattice. A necessary and sufficient condition for a sequence $(y_n)_{n=-\infty}^{\infty} \in Y$ to be positive definite is that there exists a nondecreasing function $g : T \to Y$ such that $y_n = \hat{g}(n)$ for all n.*

Proof. Assume $y_j = \hat{\boldsymbol{g}}(j)$ with $\boldsymbol{g} : T \to Y$ non-decreasing. Then

$$\sum_{m,n} c_n \overline{c_m} y_{n-m} = \int_T \Big(\sum_{m,n} c_n \overline{c_m} e^{i(n-m)t} \Big) d\boldsymbol{g}(t) = \\ = \int_T \Big| \sum_n c_n e^{int} \Big|^2 d\boldsymbol{g}(t) \geq 0 \,.$$

Conversely, if the sequence y_j is positive definite and we take $c_l = e^{ilt}$ for $|l| \leq N$, otherwise 0, then

$$\sum_{m,n} c_n \overline{c_m} y_{n-m} = (2N+1) \sigma_{2N}(Y, t) \geq 0$$

and it is enough to apply Theorem 4. □

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