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ON THE JOINT OBSERVABLE IN SOME QUANTUM STRUCTURES

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ABSTRACT. A mathematical model of quantum theory based on fuzzy sets is considered. The existence of the joint observable is proved and the sum and the independence of observables are examined.

0. Introduction

In several mathematical models of quantum theory, certain function spaces are studied instead of linear subspaces of a Hilbert space. For example a q-σ-algebra Q can be considered, i.e., a family of subsets of a set closed under complements and countable disjoint unions ([8]). Then the family $\{\chi_A; A \in Q\}$ is a function space being an orthomodular poset ([16]). In [11] some more general conditions have been examined under which a functional space forms an orthomodular poset. Recently fuzzy sets were used for constructing various models of quantum structures ([19], [17], [2]). Since a fuzzy set is a function $f\colon \Omega \to [0,1]$, the fuzzy approach leads to some functional spaces, too.

While in some preceding papers the Zadeh connectives were used (for a review see [6], [20]), in this paper we consider the Giles connectives ([18]). Of course, all necessary definitions are mentioned in the following. Further we prove that the joint observable exists, we examine the sum of observables and finally we define in a reasonable way the independence of observables.

We recall that the joint observable and the sum of observables has been introduced in [5, 7] for a model of fuzzy quantum spaces, see also [6].

For simplicity, let us consider a measurable space $(\Omega, \mathcal{S})$ and the space $\mathcal{F}$ of all measurable functions $f\colon \Omega \to [0,1]$. As usual two basic notions will be considered: state and observable.

A *state* is a mapping $m\colon \mathcal{F} \to [0,1]$ satisfying the following conditions:

(i) $m(1_\Omega) = 1$.

(ii) If $f, g, h \in \mathcal{F}$ and $f = g + h$, then $m(f) = m(g) + m(h)$.

(iii) If $f_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$, $f \in \mathcal{F}$ and $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

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Note that by Klement [9], m is an integral, i.e., there is a probability measure P on $\mathcal{S}$ such that $m(f) = \int f dP$ for any $f \in \mathcal{F}$.

An *observable* is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ ($\mathcal{B}(\mathbb{R})$ is the family of all Borel subsets of $\mathbb{R}$) satisfying the following conditions:

(i) $x(\mathbb{R}) = 1_\Omega$.
(ii) If $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.
(iii) If $A_n \in \mathcal{B}(\mathbb{R})$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Note that (i) and (ii) imply $x(A^c) = 1 - x(A)$ for any $A \in \mathcal{B}(\mathbb{R})$. It is not difficult to prove that the composite mapping $m_x\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by the formula $m_x(A) = m\big(x(A)\big)$ is a probability measure.

1. Joint observable

While an observable x corresponds to a random variable $\xi\colon \Omega \to \mathbb{R}$ (where $x(E)$ can be considered as $x(E) = \chi_{\xi^{-1}(E)})$, the joint observable corresponds to a random vector $T = (\xi, \eta)$. It can be defined as a morphism $h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ (where $h(F)$ can be considered as $h(F) = \chi_{T^{-1}(F)}$ in the crisp case). We define the joint observable of observables x and y as a mapping $h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ with the following properties:

(i) $h(\mathbb{R}^2) = 1_\Omega$.
(ii) If $A, B \in \mathcal{B}(\mathbb{R}^2)$, $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.
(iii) If $A_n \in \mathcal{B}(\mathbb{R}^2)$ $(n = 1, 2, \dots)$, $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.
(iv) $h(C \times D) = x(C) \cdot y(D)$ for every $C, D \in \mathcal{B}(\mathbb{R})$.

If we compare this definition with the definition of a random vector, we see that

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D),$$

hence

$$\chi_{T^{-1}(C \times D)} = \chi_{\xi^{-1}(C)} \chi_{\eta^{-1}(D)}.$$

Of course, this is not the only formula how to express the characteristic function $\chi_{E \cap F}$ by the help of characteristic functions χ_E, χ_F. Actually, one can use any fuzzy intersection connective, e.g., $\chi_{E \cap F} = \min(\chi_E, \chi_F)$. By [1], the only fuzzy intersection distributive with respect to the Giles fuzzy union (i.e., if $f, g, h, g + h \in F$, then $(g + h) \cap f = g \cap f + h \cap f)$, is exactly the product. This justifies using the product in (iv). Of course, we could substitute the sum by the maximum or another fuzzy union. So different fuzzy connectives lead to different fuzzy quantum models.

THEOREM 1. *For any pair of observables x, y there exists their joint observable.*

Proof. Since $x(A) \in \mathcal{F}$, $x(A)$ is a function $x(A)\colon \Omega \to [0,1]$, hence for $\omega \in \Omega$, $x(A)(\omega) \in [0,1]$. For fixed $\omega \in \Omega$ define $\mu_\omega\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ by the formula

$$\mu_\omega(A) = x(A)(\omega)\,,$$

and similarly $\nu_\omega\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ by the formula

$$\nu_\omega(B) = y(B)(\omega)\,.$$

Evidently μ_ω, ν_ω are probability measures, so we can define $h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ by the formula

$$h(C)(\omega) = \mu_\omega \times \nu_\omega(C)\,.$$

It is easy to see that h is a morphism satisfying the property

$$\begin{aligned} h(A \times B)(\omega) &= \mu_\omega \times \nu_\omega(A \times B) = \mu_\omega(A)\nu_\omega(B) = \\ &= x(A)(\omega)\, y(B)(\omega) = \big(x(A)\, y(B)\big)(\omega)\,. \end{aligned}$$

Since the equality holds for every $\omega \in \Omega$, we conclude that $h(A\times B) = x(A)y(B)$ for every $A, B \in \mathcal{B}(\mathbb{R})$. □

R e m a r k . In [14] we have shown that x is an observable (on $(\Omega, \mathcal{F})$) if and only if the mapping $\mathcal{K}\colon (\Omega, \mathcal{S}) \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ defined by $\mathcal{K}(\omega, A) = x(A)(\omega)$, is a Markov kernel, i.e., $\omega \mapsto \mathcal{K}(\omega, A)$ is an $\mathcal{S}$-measurable function for every $A \in \mathcal{B}(\mathbb{R})$ and $A \mapsto \mathcal{K}(\omega, A)$ is a probability distribution for every $\omega \in \Omega$.

Now, for fixed ω, let $\mathcal{K}_x(\omega, \cdot)$, $\mathcal{K}_y(\omega, \cdot)$ be corresponding probability distributions. Let ξ, η be independent random variables defined on Ω with probability distribution $\mathcal{K}_x(\omega, \cdot)$, $\mathcal{K}_y(\omega, \cdot)$, respectively, $T = (\xi, \eta)$, $\mathcal{K}(\omega, \cdot)$ be the probability distribution on $\mathcal{B}(\mathbb{R}^2)$ induced by T. Then $h(F)(\omega) = \mathcal{K}(\omega, F)$, $\omega \in \Omega$, $F \in \mathcal{B}(\mathbb{R}^2)$.

2. Sum of observables

Similarly as in the orthomodular poset theory, some operations with observables can be defined by the help of joint distribution. E.g.,

$$(x + y)(A) = h\big(g^{-1}(A)\big)\,,$$

where

$$g\colon \mathbb{R}^2 \to \mathbb{R}\,, \quad g(u, v) = u + v\,.$$

Indeed, if $T = (\xi, \eta)$ is a random vector, then

$$(\xi + \eta)^{-1}(A) = (g \circ T)^{-1}(A) = T^{-1}\big(g^{-1}(A)\big)\,,$$

what justifies our definition.

A natural question arises how to compute the sum of two observables. For any fixed $\omega \in \Omega$, put

$$X_\omega(t) = x\big((-\infty, t)\big)(\omega)\,.$$

Then $X_\omega\colon \mathbb{R} \to [0,1]$ is a distribution function (see [13]). This enables us to express the *sum* of observables in a very simple form (for another formula see [10], [9]).

THEOREM 2. *For every observables x, y and every $t \in \mathbb{R}$, $\omega \in \Omega$ we have*

$$(x+y)\big((\infty,t)\big)(\omega) = X_\omega * Y_\omega(t)\,,$$

where

$$X_\omega * Y_\omega(t) = \int_{-\infty}^{\infty} X_\omega(t-u)\, dY_\omega(u) = \int_{-\infty}^{\infty} Y_\omega(t-u)\, dX_\omega(u)\,.$$

Proof. Evidently

$$g^{-1}\big((-\infty,t)\big) = \big\{(u,v); u+v<t\big\} = \\ = \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \Big\langle \frac{i-1}{2^n}, \frac{i}{2^n}\Big) \times \Big(-\infty, t-\frac{i}{2^n}\Big)\,.$$

Therefore

$$h\big(g^{-1}((-\infty,t))\big) = \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=-k}^{k} x\Big(\Big\langle \frac{i-1}{2^n}, \frac{i}{2^n}\Big)\Big)\, y\Big(\Big(-\infty, t-\frac{i}{2^n}\Big)\Big)\,.$$

Then for a fixed ω we obtain

$$(x+y)\big((-\infty,t)\big)(\omega) = \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=-k}^{k} Y_\omega\Big(t-\frac{i}{2^n}\Big)\Big(X_\omega\Big(\frac{i}{2^n}\Big) - X_\omega\Big(\frac{i-1}{2^n}\Big)\Big) = \\ = \int_{-\infty}^{\infty} Y_\omega(t-u)\, dX_\omega(u)\,.$$

□

Remarks. 1. Note that the sum of observables introduced in [10] (and similarly in [9]) corresponds to the Menger approach to the combination of two distributions ([12]). Our approach leads to the Wald combination ([21]).

2. Taking into account the results of Alsina ([1]), the only convenient fuzzy connectives ensuring $(A \cap B) \cup (A \cap B^c) = A$ for any $A, B \in \mathcal{T}$, are induced by a t-norm T, t-conorm S and complementation c given by

$T(a,b) = \varphi^{-1}\big(\varphi(a)\cdot\varphi(b)\big)$,

$S(a,b) = \varphi^{-1}\big(\min(1, \varphi(a)+\varphi(b))\big)$,

$C(a) = \varphi^{-1}\big(1-\varphi(a)\big)$,

for any $a, b \in [0,1]$. Here $\varphi\colon [0,1] \to [0,1]$ is any given continuous, strictly increasing mapping, $\varphi(0) = 0$, $\varphi(1) = 1$. If we put $\varphi(t) = t$, $t \in [0,1]$, we get exactly the fuzzy connectives used throughout the paper. We see that this is exactly the only possible case up to the isomorphism φ.

3. Independence

The problem arises what to do if the domain $\mathcal{F}$ is not closed under the product of functions, or if the domain is more abstract. As a motivation we mention MV algebras ([3]). An elementary example of an MV algebra is a picture whose darkness can be interpreted as a fuzzy set $f\colon \Omega \to [0,1]$. If we have two pictures $f, g\colon \Omega \to [0,1]$, then their composition can be described as the fuzzy set

$$f \oplus g = \max(f+g, 1)\,,$$

because the degree of darkness of a colour cannot exceed the maximum 1 (1 corresponds to the black, 0 to the white). MV algebras have important connections with C^*-algebras, fuzzy sets, l-groups and multivalued logics. E.g., every MV-algebra is isomorphic to an interval in an l-group, i.e., our starting example with pictures is a typical one in some sense ([15]).

Generally we shall suppose only that a partially ordered set $\mathcal{F}$ is given with the greatest element 1 and with a partial commutative binary operation $\oplus$ such that if $f \oplus g$ and $f \oplus h$ are defined, then $g \leqq h$ implies $f \oplus g \leqq f \oplus h$. Again a state is a mapping $m\colon \mathcal{F} \to [0,1]$ satisfying the following conditions:

(i) $m(1) = 1$,
(ii) $f = g \oplus h \Rightarrow m(f) = m(g) + m(h)$,
(iii) $f_n \nearrow f \Rightarrow m(f_n) \nearrow m(f)$.

An observable is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(\mathbb{R}) = 1$,
(ii) $A \cap B = \emptyset \Rightarrow x(A \cup B) = x(A) \oplus x(B)$,
(iii) $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$.

Again the composite mapping $m_x\colon \mathcal{B}(\mathbb{R}) \to [0,1]$, $m_x(E) = m\big(x(E)\big)$, is a probability measure for arbitrary binary operation $\oplus$.

The joint observable of two observables x, y should be a mapping

$$h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$$

being a morphism and satisfying the identity

$$h(A \times B) = x(A) \odot y(B)\,,$$

where $\odot$ is some binary operation of $\mathcal{F}$. It may happen, in general, that such h cannot be defined. Another situation is in the case of independent observables, what is an important case in the corresponding probability considerations. Usually the independence is defined by the formula

$$m\big(x(A) \odot y(B)\big) = m\big(x(A)\big)\, m\big(y(B)\big)\,.$$

Combining these two formulas we obtain the following general definition.

DEFINITION. Two observables $x, y\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ are said to be *independent*, in the state m if there exists a morphism $h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ such that

$$m\big(h(A \times B)\big) = m\big(x(A)\big)\, m\big(y(B)\big)$$

for every $A, B \in \mathcal{B}(\mathbb{R})$.

If $\mathcal{F}$ is the set of all measurable functions from Ω to $\langle 0,1\rangle$, then the independence has the usual meaning:

$$\int f\,g\,dP = \int f\,dP \int g\,dP$$

for every $f \in x\big(\mathcal{B}(\mathbb{R})\big)$, $g \in y\big(\mathcal{B}(\mathbb{R})\big)$. On the other hand the notion of independent observables can be considered in more general situations, too. Consider, e.g., the q-algebra $Q = \big\{\emptyset, \{1,2\}, \{3,4\}, \{1,3\}, \{2,4\}, \{1,2,3,4\}\big\}$ of subsets of $\{1,2,3,4\}$, $\mathcal{F} = \{\chi_A; A \in Q\}$. For $f, g \in \mathcal{F}$ define $f \oplus g = f + g$, if $f + g \leqq 1$. Further put

$$\begin{aligned} m\big(\chi_{\{1,2\}}\big) &= m\big(\chi_{\{3,4\}}\big) = \frac{1}{2}\,, \\ m\big(\chi_{\{2,4\}}\big) &= m(0) = 0\,, \\ m\big(\chi_{\{1,3\}}\big) &= m(1) = 1\,. \end{aligned}$$

It is easy to see that $m\colon \mathcal{F} \to [0,1]$ is a state. Define now two observables $x, y\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ by the following way.

$$x(A) = \begin{cases} 1, & \text{if } 0 \in A,\ 1 \in A, \\ \chi_{\{1,2\}}, & \text{if } 0 \in A,\ 1 \notin A, \\ \chi_{\{3,4\}}, & \text{if } 0 \notin A,\ 1 \in A, \\ 0, & \text{if } 0 \notin A,\ 1 \notin A, \end{cases}$$

$$y(B) = \begin{cases} 1, & \text{if } 0 \in B,\ 1 \in B, \\ \chi_{\{1,3\}}, & \text{if } 0 \in B,\ 1 \notin B, \\ \chi_{\{2,4\}}, & \text{if } 0 \notin B,\ 1 \in B, \\ 0, & \text{if } 0 \notin B,\ 1 \notin B. \end{cases}$$

We shall show that x, y are independent, although $\mathcal{F}$ is not closed under the product of functions. Define hence

$$h(C) = \begin{cases} 1, & \text{if } (0,0) \in C,\ (1,0) \in C, \\ \chi_{\{1,2\}}, & \text{if } (0,0) \in C,\ (1,0) \notin C, \\ \chi_{\{3,4\}}, & \text{if } (0,0) \notin C,\ (1,0) \in C, \\ 0, & \text{if } (0,0) \notin C,\ (1,0) \notin C. \end{cases}$$

Then $h\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ is a morphism and $h(A\times B) = x(A)$, if $0 \in B$, $h(A\times B) = 0$, if $0 \notin B$. Therefore

$$m\big(h(A \times B)\big) = m\big(x(A)\big)\, m\big(y(B)\big)$$

whenever $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathbb{R})$.

Again we can define the sum of x and y by the formula

$$(x+y)(A) = h\big(g^{-1}(A)\big)\,,$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$, $g(u,v) = u + v$ and we can ask about the distribution $m\big((x+y)(-\infty,t)\big)$. The answer is the following.

THEOREM 3. *Let x and y be two independent observables. Then*

$$m\big((x+y)(-\infty,t)\big) = m_x \times m_y\big(\{(u,v); u+v < t\}\big)$$

for every $t \in \mathbb{R}$.

P r o o f. We know that

$$g^{-1}\big((-\infty,t)\big) = \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \Big\langle \frac{i-1}{2^n}, \frac{i}{2^n}\Big) \times \Big(-\infty, t - \frac{i}{2^n}\Big)\,,$$

hence

$$\begin{aligned} m\Big(h\big(g^{-1}((-\infty,t))\big)\Big) &= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} m\Big(x\big(\big\langle\frac{i-1}{2^n}, \frac{i}{2^n}\big)\big)\Big) m\Big(y\big((-\infty, t - \frac{i}{2^n})\big)\Big) = \\ &= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} m_x \times m_y\Big(\big\langle\frac{i-1}{2^n}, \frac{i}{2^n}\big) \times \big(-\infty, t - \frac{i}{2^n}\big)\Big) = \\ &= m_x \times m_y\big(g^{-1}(-\infty,t)\big)\,. \end{aligned}$$

□

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