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AN ABSTRACT FORM OF A CONDITIONAL CAUCHY'S EQUATION

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Let D and E be non-void sets, let $\varphi\colon D \to D$, $\psi\colon D \to D$ and $\Phi\colon E \times E \to E$ be maps. The functional equation

$$\Phi(f(x), f(\varphi(x)) = f(\psi(x))$$

with one unknown function $f\colon D \to E$ is investigated and a problem of K. Lajkó is discussed.

The aim of the present paper is to give a general solution of an abstract functional equation $\Phi\big(f(x), f(\varphi(x))\big) = f\big(\psi(x)\big)$ with one unknown function f. The investigation of the above functional equation has been motivated by the following problem of K. Lajkó formulated in [1] (Problem P214): Under what conditions is $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}_+$ ($\mathbb{R}_+$ – the set of positive reals), the only solution of

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

for $x \in \mathbb{R}_+$? Note, that the Lajkó equation can be considered as a conditional Cauchy's functional equation $f(x+y) = f(x) + f(y)$ postulated only for x and y satisfying $xy = 1$.

We shall show, under some assumptions, that any solution of the abstract functional equation is uniquely determined by its initial conditions.

Let D and E be non-void sets, let $\Phi\colon E \times E \to E$. Further, let φ and ψ be transformations of the set D, i.e., $\varphi\colon D \to D$ and $\psi\colon D \to D$. We shall deal with the functional equation

$$\Phi\big(f(x), f(\varphi(x))\big) = f\big(\psi(x)\big) \tag{1}$$

with one unknown function $f\colon D \to E$.

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THEOREM 1. *Let sets D, E and the functions Φ, φ, ψ have the introduced meaning. Let*

(i) $D = D_0 \cup D_1 \cup D_2$, $D_1 \neq \emptyset$, $D_2 \neq \emptyset$, $D_i \cap D_j = \emptyset$ *for* $i \neq j$, *such that* $\varphi(D_0) \subset D_0$, $\varphi(D_1) \subset D_2$, $\varphi(D_2) \subset D_1$ *and* $\psi(D_0) \subset D_0 \cup D_2$, $\psi(D_1 \cup D_2) \subset D_2$;

(ii) $\varphi \circ \varphi = I$ *(I – the identity transformation of D),* $\psi \circ \varphi = \psi$;

(iii) Φ *be a symmetric function, i.e.,* $\Phi(X,Y) = \Phi(Y,X)$ *whenever* $X,Y \in E$.

Let $f_0\colon D_0 \cup \psi(D_0) \to E$ *be a solution of* (1) *on* D_0, *i.e.,* (1) *holds for every* $x \in D_0$. *Let* $g\colon D_2 \to E$ *be an arbitrary function with* $g(x) = f_0(x)$ *for each* $x \in D_2 \cap \psi(D_0)$. *Further let*

(iv) $\Psi\colon g(D_2) \times g(D_2) \to E$ *be a function such that* $Y = \Psi(X,Z)$ *implies* $Z = \Phi(X,Y)$ *for every* $X, Z \in g(D_2)$ *and* $Y \in E$.

Then $f\colon D \to E$ *given by*

$$f(x) = \begin{cases} = f_0(x), & \text{for } x \in D_0; \\ = \Psi\big(g(\varphi(x)), g(\psi(x))\big) & \text{for } x \in D_1; \\ = g(x) & \text{for } x \in D_2 \end{cases} \tag{2}$$

is a solution of (1) *with* $f \mid D_0 \cup \psi(D_0) = f_0$.

Proof. If $f\colon D \to E$ is determined by (2), then obviously $f \mid D_0 \cup \psi(D_0) = f_0$.

Suppose $x \in D_0$. Then $\varphi(x) \in D_0$, $\psi(x) \in D_0 \cup \psi(D_0)$ and $\Phi\big(f(x), f(\varphi(x))\big) = \Phi\big(f_0(x), f_0(\varphi(x))\big) = f_0\big(\psi(x)\big) = f\big(\psi(x)\big)$.

Suppose $x \in D_1$. Then $\varphi(x) \in D_2$, $\psi(x) \in D_2$ and $f(x) = \Psi\big(g(\varphi(x)), g(\psi(x))\big)$. Hence $f\big(\psi(x)\big) = g\big(\psi(x)\big) = \Phi\big(g(\varphi(x)), f(x)\big) = \Phi\big(f(x), f(\varphi(x))\big)$.

Suppose $x \in D_2$. Then $\varphi(x) \in D_1$, $\psi(x) \in D_2$ and $f\big(\varphi(x)\big) = \Psi\big(g(\varphi(\varphi(x))), g(\psi(\varphi(x)))\big) = \Psi\big(g(x), g(\psi(x))\big)$. Hence $f\big(\psi(x)\big) = g\big(\psi(x)\big) = \Phi\big(g(x), f(\varphi(x))\big) = \Phi\big(f(x), f(\varphi(x))\big)$. □

Remark. Theorem 1 asserts that an arbitrary function $g\colon D_2 \to E$ (with $g(x) = f_0(x)$ for $x \in D_2 \cap \psi(D_0)$) by means of (2) gives a solution of the equation (1). Because the function Ψ of the condition (iv) is not uniquely determined, we need not obtain, by this way, all solutions of (1). This shows the following example.

EXAMPLE. Let us consider the functional equation

$$f(x)^2 + f(1-x)^2 = f\big(x^3(1-x)^3\big), \tag{3}$$

$f\colon (0,1) \to \mathbb{R}$ ($\mathbb{R}$ – the real line). Put $D_0 = \{\frac{1}{2}\}$, $D_1 = (\frac{1}{2}, 1)$, $D_2 = (0, \frac{1}{2})$, $\varphi(x) = 1-x$, $\psi(x) = x^3(1-x)^3$, $Z = \Phi(X,Y) = X^2+Y^2$ and $Y = \Psi(X,Z) = \sqrt{-X^2+Z}$. Obviously $D_0 \cup \psi(D_0) = \{\frac{1}{64}, \frac{1}{2}\}$. If we suppose $f_0\left(\frac{1}{2}\right) = \frac{1}{2}$, then (3) implies $f_0\left(\frac{1}{64}\right) = \frac{1}{2}$. Put $g(x) = \frac{1}{2}$ for $x \in (0, \frac{1}{2})$. Then $g(D_2) = \{\frac{1}{2}\}$ and it is easy to verify that the assumptions of Theorem 1 are fulfilled. It follows from (2) that $f(x) = \frac{1}{2}$ for every $x \in (0,1)$. The functional equation (3) has still another solution $f^*\colon (0,1) \to \mathbb{R}$ (with $f^* \mid D_2 = g$) belonging to the function $Y = \Psi^*(X,Z) = -\sqrt{-X^2+Z}$: $f^*(x) = \frac{1}{2}$ for $x \in (0, \frac{1}{2}]$, $f^*(x) = -\frac{1}{2}$ for $x \in (\frac{1}{2}, 1)$.

The assumptions of Theorem 1 can be strengthened in such a way, that each solution of (1) can be obtained by a suitable choice of the function g.

THEOREM 2. *Let the sets D, E and the functions Φ, φ, ψ have the introduced meaning. Let the assumptions* (i), (ii), *and* (iii) *of Theorem 1 be fulfilled and let the functions f_0 and g have the meaning of Theorem 1. Further let*

(iv*) $\Psi\colon g(D_2) \times g(D_2) \to E$ *be a function such that for every X, $Z \in g(D_2)$ and $Y \in E$ $\quad Y = \Psi(X,Z)$ if and only if $Z = \Phi(X,Y)$.*

Then $f\colon D \to E$ given by relations (2) *is a solution of* (1) *with the property $f \mid D_0 \cup \psi(D_0) = f_0$, and, every solution of* (1) *with this property can be obtained using relations* (2) *and a suitable choice of the function $g\colon D_2 \to E$.*

P r o o f. Let f_1 be a solution of (1). It is sufficient to show that every solution f of (1) with $f \mid D_0 \cup \psi(D_0) = f_0 = f_1 \mid D_0 \cup \psi(D_0)$ and $f \mid D_2 = g = f_1 \mid D_2$ is given by relations (2) and $f(x) = f_1(x)$ holds for every $x \in D$.

If $x \in D_0 \cup D_2$, then relations (2) and $f(x) = f_1(x)$ obviously hold. Suppose $x \in D_1$. Since (1) is fulfilled, we have $f(\psi(x)) = \Phi(f(\varphi(x)),\, f(x))$, $f(x) = \Psi(f(\varphi(x)),\, f(\psi(x))) = \Psi(g(\varphi(x)),\, g(\psi(x)))$ hence relations (2) are fulfilled. We prove $f(x) = f_1(x)$ for $x \in D_1$. The equality $\Phi(f_1(x),\, f_1(\varphi(x))) = f_1(\psi(x))$ is equivalent to $\Psi(f_1(\varphi(x)),\, f_1(\psi(x))) = f_1(x)$. Obviously $f(x) = \Psi(g(\varphi(x)),\, g(\psi(x))) = \Psi(f_1(\varphi(x)),\, f_1(\psi(x)))$. Consequently $f(x) = f_1(x)$. □

EXAMPLES. We shall give a condition such that a quadratic function $f(x) = ax^2 + bx + c$ is the only solution of

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right), \tag{4}$$

$f\colon D \to \mathbb{R}$, $D \subset \mathbb{R}$.

a) Suppose $D = (0,1) \cup (1,\infty)$. Put $D_0 = \emptyset$, $D_1 = (0,1)$, $D_2 = (1,\infty)$, $\varphi(x) = \frac{1}{x}$, $\psi(x) = x + \frac{1}{x}$, $\Phi(X,Y) = X + Y$ and $\Psi(X,Z) = -X + Z$. It is easy to verify that the assumptions of Theorem 2 are fulfilled. Obviously necessary $g(x) = ax^2 + bx + c$ holds for $x \in (1, \infty)$. The relations (2) imply that the solution of (4) is of the form $f(x) = ax^2 + bx + 2a$ for $x \in (0,1)$ and $f(x) = ax^2 + bx + c$ for $x \in (1, \infty)$. Consequently, the equation (4) has on $D = (0,1) \cup (1, \infty)$ the only solution a quadratic function f if and only if $g = f \mid (1, \infty)$ is a quadratic function of the form $g(x) = ax^2 + bx + 2a$, where a, b are arbitrary real constants.

b) Suppose $D = (0, \infty)$. Put $D_0 = \{1\}$ and D_1, D_2, φ, ψ, Φ, Ψ as in the case a). The assumptions of Theorem 2 are fulfilled and $D_0 \cup \psi(D_0) = \{1, 2\}$. If $f_0(1) = k$, then (4) implies $f_0(2) = 2k$. Hence every solution f of (4) on $(0, \infty)$ (with $f(1) = k$) is determined by a function $g\colon (1, \infty) \to \mathbb{R}$ with $g(2) = 2k$ and by relations (2). A simple computation gives that (4) has on $D = (0, \infty)$ the only solution a quadratic function f if and only if $g = f \mid (1, \infty)$ is a quadratic function of the form $g(x) = ax^2 + (k - 3a)x + 2a$, where $k = f_0(1)$ and a is an arbitrary real constant.

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