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ON F–CONTINUITY OF REAL FUNCTIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. The concept of F–continuity of real numbers is based on the well-known notion of almost convergence of sequences of real numbers. From the F–continuity of a function at a point its linearity follows. This fact strengthens a result of E. Ö z t ü k [7].

Introduction

In paper [7] the notion of almost continuity of real functions is introduced. This notion is based on the concept of F-convergence from [6] (see also [8], p. 59–60 and [9]). Since the term "almost continuity" has a different meaning in the theory of real functions (cf. [5], [11]), we shall use the notion F–continuity instead of almost continuity from [7].

Definitions and Notations

A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be *almost convergent* (or F-convergent) to the number s if

$$\lim_{p\to\infty} \frac{x_{n+1} + \cdots + x_{n+p}}{p} = s \tag{1}$$

holds uniformly in $n = 0, 1, \dots$ (cf. [6], [8] p. 59–60, [9]). If (1) holds uniformly in $n = 0, 1, \dots$, then s is uniquely determined and we write $F\text{–}\lim x_n = s$ (see [6]).

If $(x_n)_{n=1}^{\infty}$ is a convergent sequence (in the usual sense), then it is also almost convergent and $F\text{–}\lim x_n = \lim\limits_{n\to\infty} x_n$.

It is well-known that each almost convergent sequence is bounded (cf. [6]).

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DEFINITION 1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *F-continuous at* $x_0 \in \mathbb{R}$ if

$$F\text{–}\lim x_n = x_0 \implies F\text{–}\lim f(x_n) = f(x_0)\,.$$

If f is F–continuous at each $x \in \mathbb{R}$, then f is called *F-continuous on* $\mathbb{R}$.

In what follows $y_n \to y$ $(n \to \infty)$ denotes the usual convergence of the sequence $(y_n)_{n=1}^{\infty}$ to the number y.

The concept of F–continuity of functions is very similar to the concept of C–continuity of functions (cf. [10]). We recall that a sequence $(x_n)_{n=1}^{\infty}$ is said to be $(C,1)$–summable to $s \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = s \tag{2}$$

(we write for brevity $(C,1)-\lim x_n = s$). A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *C-continuous* at $x_0 \in \mathbb{R}$ if

$$(C,1)\text{–}\lim x_n = x_0 \implies (C,1)\text{–}\lim f(x_n) = f(x_0)\,.$$

A function f is said to be *C-continuous* on $\mathbb{R}$ if it is C–continuous at each $x \in \mathbb{R}$.

It is proved in the solution of the Problem in [10] that if f is C–continuous at a point $x_0 \in \mathbb{R}$, then it is linear. The main aim of this note is to prove an analogous result for F–continuity of functions.

Main Results

In [7] the following result (formulated in our terminology) is proved (cf. [7], Theorem (3.2)):

THEOREM A. *Let a function $f\colon \mathbb{R} \to \mathbb{R}$ be F–continuous at $x_0 \in \mathbb{R}$. Then f is continuous at x_0 if and only if*

$$f(x_{n+1}) - f(x_n) \to 0 \quad (n \to \infty)$$

for each sequence $(x_n)_{n=1}^{\infty}$ converging to x_0.

We shall strengthen this result by showing that each function $f\colon \mathbb{R} \to \mathbb{R}$ which is F–continuous at a point is already linear on $\mathbb{R}$, i.e. it has the form $f(x) = a\,x + b$, where a, b are constants.

Hence we shall prove the following

THEOREM 1. *If $f\colon \mathbb{R} \to \mathbb{R}$ is F–continuous at a point $x_0 \in \mathbb{R}$, then f is a linear function.*

Remark 1. It can be easily checked that each linear function is F–continuous at every point $x \in \mathbb{R}$. Consequently Theorem 1 says that from F–continuity of a function at a single point its F–continuity on $\mathbb{R}$ follows. If $f\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function and C_f^* is the set of all points at which f is F–continuous, then we have only the following two possibilities:

a) $C_f^* = \emptyset$ b) $C_f^* = \mathbb{R}$.

In the proof of Theorem 1 we shall use some ideas from the solution of Problem 4216 (cf. [10]) and the following lemma.

LEMMA 1. *If $f\colon \mathbb{R} \to \mathbb{R}$ is F–continuous at $x_0 \in \mathbb{R}$, then it is continuous at x_0.*

Proof. First of all we prove that the function f is bounded at x_0, i.e., that there exists a $d > 0$ such that f is bounded on the interval $(x_0 - d, x_0 + d)$. To prove this it suffices to show that if $x_n \to x_0$ $(n \to \infty)$, then the sequence $\big(f(x_n)\big)_{n=1}^{\infty}$ is bounded.

Let $x_n \to x_0$ $(n \to \infty)$. Then $F\text{–}\lim x_n = x_0$ and by the assumption of Lemma we have $F\text{–}\lim f(x_n) = f(x_0)$. Hence $\big(f(x_n)\big)_{n=1}^{\infty}$ as an almost convergent sequence is bounded.

We now can prove the continuity of the function f at the point x_0. Suppose that f is discontinuous at x_0. Since it is bounded on an interval $(x_0 - d, x_0 + d)$ $(d > 0)$, there exists a sequence $(y_n)_{n=1}^{\infty}$ of elements of $(x_0 - d, x_0 + d)$ such that $y_n \to x_0$ $(n \to \infty)$ and $\big(f(y_n)\big)_{n=1}^{\infty}$ converges to a number $b \neq f(x_0)$. From this we get

$$F\text{–}\lim f(y_n) = b\,. \tag{3}$$

On the other hand from $y_n \to x_0$ $(n \to \infty)$ we have $F\text{–}\lim y_n = x_0$ and so by the assumption of Lemma we get

$$F\text{–}\lim f(y_n) = f(x_0) \neq b\,.$$

This contradicts (3). The proof is finished. □

Remark 2. It follows from Theorem 1 that Lemma 1 cannot be conversed.

Proof of Theorem 1. First of all we shall prove the following special case of Theorem 1. We shall assume that $g\colon \mathbb{R} \to \mathbb{R}$ is F–continuous at the point 0 and $g(0) = 0$.

Let a, b, c be real numbers such that $a + b + c = 0$. Construct the sequence

$$(x_n)_{n=1}^{\infty} = a, b, c, a, b, c, \dots\,.$$

We show that this sequence is almost convergent to 0. Indeed, put

$$A_{n,p} = \frac{x_{n+1} + \cdots + x_{n+p}}{p} = \frac{B_{n,p}}{p} \quad (n, p \in N).$$

Assume that $p > 3$. Then p can be expressed in the form $p = 3q + r$, $q \geqq 1$, $0 \leqq r \leqq 2$, q, r are integers.

If $x_{n+1} = a$, then for $B_{n,p}$ we have the following possibilities: $B_{n,p} = q(a + b + c) = 0$ (if $r = 0$), $B_{n,p} = q(a + b + c) + a = a$ (if $r = 1$) and $B_{n,p} = q(a + b + c) + a + b = a + b$ (if $r = 2$).

If $x_{n+1} = b$, then $B_{n,p} = (q - 1)(a + b + c) + (b + c) + a = 0$ (if $r = 0$), $B_{n,p} = (q - 1)(a + b + c) + (b + c) + a + b = b$ (if $r = 1$) and $B_{n,p} = (q - 1)(a + b + c) + (b + c) + (a + b + c) = b + c$ (if $r = 2$).

Finally if $x_{n+1} = c$, then $B_{n,p} = c + (q - 1)(a + b + c) + a + b = 0$ (if $r = 0$), $B_{n,p} = c + q(a + b + c) = c$ (if $r = 1$), $B_{n,p} = c + q(a + b + c) + a = a + c$ (if $r = 2$).

Hence $A_{n,p}$ has one of the values:

$$\frac{0}{p}, \frac{a}{p}, \frac{a+b}{p}, \frac{b}{p}, \frac{b+c}{p}, \frac{c}{p}, \frac{a+c}{p} \quad \text{for each} \quad n \geqq 0$$

and $p > 3$. From this it is obvious that the $A_{n,p}$ by $p \to \infty$ converges to 0 uniformly in $n = 0, 1, \dots$. According to the assumption of Theorem we have $F\text{–}\lim g(x_n) = g(0) = 0$ i.e. the sequence

$$\big(g(x_n)\big)_{n=1}^{\infty} = g(a),\ g(b),\ g(c),\ g(a),\ g(b),\ g(c), \dots$$

is almost convergent to 0. But a direct calculation shows that

$$F\text{–}\lim g(x_n) = \frac{g(a) + g(b) + g(c)}{3}.$$

Hence

$$g(a) + g(b) + g(c) = 0. \tag{4}$$

Since $c = -a - b$, we get $g(-a - b) = -g(a) - g(b)$.
Putting $b = 0$ we have

$$g(-a) = -g(a) \quad (a \in \mathbb{R}). \tag{5}$$

Let $x, y \in \mathbb{R}$ be arbitrary. Put $c = x + y$, $a = -x$, $b = -y$. Then $a + b + c = 0$ and according to (4) and (5) we get $g(x + y) = -g(-x) - g(-y) = g(x) + g(y)$.

Hence the function $g\colon \mathbb{R} \to \mathbb{R}$ satisfies the Cauchy functional equation $g(x+y) = g(x) + g(y)$ and according to Lemma 1 it is continuous at 0. On the basis of the well-known knowledge on Cauchy equation we get $g(x) = a\,x$ for $x \in \mathbb{R}$, a being a constant (cf. [3], p. 44–45).

We shall now discuss the general case. Let $f\colon \mathbb{R} \to \mathbb{R}$ be F–continuous at a point $x_0 \in \mathbb{R}$. We introduce new coordinates $x' = x - x_0$, $y' = y - f(x_0)$. Put $g(x') = f(x) - f(x_0)$. It is easy to verify that from the F–continuity of f at x_0 the F–continuity of g at 0 follows. On the basis of the previous part of the proof the function g has the form $g(x') = a\,x'$, i.e. $f(x) - f(x_0) = a(x - x_0) = a\,x - a\,x_0$, $f(x) = a\,x + \big(f(x_0) - ax_0\big) = a\,x + b$, where $b = f(x_0) - a\,x_0$. The proof is finished. □

Considerations about C–continuity can be extended for arbitrary regular summation matrix A instead of the Cesàro matrix $(c_{nk})(c_{nk} = \frac{1}{n}$ $(k = 1, 2, \dots, n)$ and $c_{nk} = 0$ for $k > n)$. Such extensions of C–continuity are discussed in papers [1] and [2]. The A–continuity of a function $f\colon \mathbb{R} \to \mathbb{R}$ can be defined in a similar manner as the C–continuity. Let A–$\lim y_n$ denote the number to which the sequence $(y_n)_{n=1}^{\infty}$ is summed by the matrix A. A function f is said to be *A–continuous* at $x_0 \in \mathbb{R}$ if

$$A\text{–}\lim x_n = x_0 \implies A\text{–}\lim f(x_n) = f(x_0)\,.$$

A function f is said to be *A–continuous* on $\mathbb{R}$ if it is A–continuous at each $x \in \mathbb{R}$.

In the papers [1] and [2] sufficient conditions are given for from A–continuity of f on $\mathbb{R}$ or A–continuity of f at a single point the linearity of f follows. These conditions concern a wide class of regular matrices. Let us remark that almost convergence is not equivalent to *A-summability* where A is an arbitrary regular matrix. (Therefore our Theorem 1 is not a consequence of the mentioned results from [1] and [2] concerning the A–continuity). If namely the summability method given by A is equivalent to the convergence, then F–$\lim x_n$ exists for $x_n = (-1)^{n-1}$ $(n = 1, 2, \dots)$, but A–$\lim x_n$ does not exists and if the method given by A is stronger than the convergence, then according to the well-known theorem of M a z u r and O r l i c z (cf. [4], p. 375–376) at least one unbounded sequence $(y_n)_1^{\infty}$ is A-summable, but F–$\lim y_n$ cannot exist since F-convergent sequences are bounded (cf. [6]). Hence the convergence fields of the method F of almost convergence and of the method given by A cannot coincide.

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