Tatra Mountains Math. Publ., 2 (1993), 105–121

TATRA MOUNTAINS
Mathematical Publications

A WEAKENING OF THE ATTOUCH–WETS TOPOLOGY ON FUNCTION SPACES

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Recently a new hyperspace topology with good applications in minimization problems was introduced. This topology is called bounded proximal topology. We study properties of the bounded proximal topology and the upper Attouch–Wets topology applied on function spaces. These topologies exhibit a good behaviour on a continuous dual X^* of a linear metrizable space X.

1. Introduction

Recently a new hyperspace topology with good applications in minimization problems was introduced. This topology is called bounded proximal topology [BL1], [BL2]. The bounded proximal topology arose as a weakening of the Attouch–Wets topology with good behaviour in many directions.

Attempts to obtain a suitable infinite dimensional generalization of Kuratowski convergence of sequences of closed convex sets [Ku] (which work good in finite dimensions) led to the notions of Mosco convergence and Attouch–Wets convergence [Be2], [Be3], [Be4], [Mo]. Unlike Mosco convergence, Attouch–Wets convergence seems to carry over all properties of Kuratowski convergence in finite dimensions to infinite dimensions [Be5].

Since the Attouch–Wets topology is much stronger than the Mosco topology it seems to be suitable to work with the intermediate bounded proximal topology.

It is the purpose of this article to exhibit a behaviour of the bounded proximal topology on the space of continuous functions and mainly on the continuous dual of linear metrizable space. As an application we see that in normed linear spaces the bounded proximal convergence of a sequence of closed hyperplanes to a closed hyperplane is equivalent to the Attouch–Wets convergence of the sequence.

AMS Subject Classification (1991): 54C35, 54B20.
Key words: Attouch–Wets topology, bounded proximal topology, continuous dual.

2. Preliminaries

(X,d) will denote a metrizable space X with a compatible metric d. The open (resp. closed) d-ball with center $x_0 \in X$ and radius $\varepsilon > 0$ will be denoted by $S_d[x_0,\varepsilon]$ (resp. $B_d[x_0,\varepsilon]$) and the ε-parallel body

$$\cup\{S_d[a,\varepsilon] :\ a \in A\}$$

for a subset A of X will be denoted by $S_d[A,\varepsilon]$.

Let $CL(X)$ be the family of all nonempty closed subsets of (X,d) and $CLB(X)$ be the family of all nonempty closed and bounded subsets of (X,d). If $A \in CL(X)$, the distance functional $d(\cdot,A) :\ X \to [0,\infty)$ is described by the familiar formula

$$d(x,A) = \inf\{d(x,a) :\ a \in A\}.$$

The gap $D_d(A,B)$ between two closed sets A and B is defined by the following formula

$$D_d(A,B) = \inf\{d(a,b) :\ a \in A,\ b \in B\}.$$

For $E \subset X$, we specify the following subsets of $CL(X)$:

$$E^- = \{F \in CL(X) :\ F \cap E \neq \emptyset\}, \qquad E^+ = \{F \in CL(X) :\ F \subset E\},$$

$$E^{++} = \{F \in CL(X) :\ \text{there is } \varepsilon > 0 \text{ such that } S_d[F,\varepsilon] \subset E\}.$$

In [BL1] was shown that all of the standard hyperspace topologies arise as weak topologies generated by families of geometric functionals defined on closed sets. The natural example is the Wijsman topology $\tau_W(d)$ [Bel] on $CL(X)$, which is the weakest one such that for each $x \in X$, the function $A \to d(x,A)$ is continuous on $CL(X)$. Also the bounded proximal topology σ_d which will be mainly dealt in this paper can be described in this sense

Definition (**[BL2]**). Let (X,d) be a metric space. The *bounded proximal topology* σ_d is the weakest topology τ on $CL(X)$ such that for each $B \in CLB(X)$ the gap functional $A \to D_d(B,A)$ is τ- continuous on $CL(X)$.

Thus σ_d is completely regular and since σ_d is finer than the Wijsman topology on $CL(X)$, which is Hausdorff, σ_d is also Hausdorff. The topology σ_d is also weaker than the proximal topology, the weak topology determined by

$$\{D_d(F,\cdot) :\ F \in CL(X)\}$$

[BLLN].

We will mainly use the local presentation of the topology σ_d [BL1]:

THEOREM A. *The bounded proximal topology* σ_d *on* $CL(X)$ *has as a local base at* $A \in CL(X)$ *all sets of the form*

$$\Phi_A[n,\ a_1,\ a_2, \dots a_k] = \{F \in CL(X)\colon\ F \cap S_d[x_0, n] \subset S_d\left[A, \tfrac{1}{n}\right], \text{ and for each } i \le k\ d(a_i, F) < \tfrac{1}{n}\},$$

where x_0 *is a fixed but arbitrary point of* X, $\{a_1,\ a_2, \dots a_k\}$ *is a finite subset of* A *and* $n \in \mathbb{Z}^+$.

THEOREM B. *Let* (X, d) *be a metric space. A subbase for* σ_d *consists of all sets of the form* V^-, *where* V *is open in* X, *and all sets of the form* $(B^c)^{++}$, *where* $B \in CLB(X)$ *and* B^c *is the complement of* B.

We shall denote by $\tau_{AW}(d)$ the metrizable topology on $CL(X)$ of uniform convergence of distance functionals on bounded subsets of X corresponding to a fixed metric d on X (the Attouch–Wets topology). A local base for $\tau_{AW}(d)$ at $A \in CL(X)$ [BDC] consists of all sets of the form

$$\{F \in CL(X) :\ \sup_{x \in B} |d(x, F) - d(x, A)| < \varepsilon\},$$

where $B \in CLB(X)$ and $\varepsilon > 0$.

Another local base for $\tau_{AW}(d)$ at $A \in CL(X)$ [Be2], [BDC] consists of all sets of the form

$$\Sigma_n[A] = \{F \in CL(X)\colon\ F \cap S_d[x_0, n] \subset S_d\left[A, \tfrac{1}{n}\right] \quad \text{and} \quad A \cap S_d[x_0, n] \subset S_d\left[F, \tfrac{1}{n}\right]\},$$

where x_0 is a fixed but arbitrary point from X.

It is very easy to see from the local presentations of σ_d and $\tau_{AW}(d)$ that $\sigma_d \subset \tau_{AW}(d)$.

The Attouch–Wets topology splits into its lower and upper halves [BL1] $\tau^+_{AW}(d)$ and $\tau^-_{AW}(d)$ where a local base for $\tau^+_{AW}(d)$ (resp. $\tau^-_{AW}(d)$) at A consists of all sets of the form

$$\Theta^+_A[B, \varepsilon] = \{F \in CL(X)\colon\ \text{for each } x \in B \quad d(x, A) - \varepsilon < d(x, F)\}$$

$$(\Theta^-_A[B, \varepsilon] = \{F \in CL(X)\colon\ \text{for each } x \in B \quad d(x, F) < d(x, A) + \varepsilon\}),$$

where $B \in CLB(X)$ and $\varepsilon > 0$.

We will see further that the upper half of the Attouch–Wets topology applied on function spaces exhibits reasonable properties under some conditions on spaces. However, $\tau^+_{AW}(d)$ on $CL(X)$ behaves badly; if X is a metric space with at least two different points, then $\tau^+_{AW}(d)$ on $CL(X)$ is not T_1-space; $\tau^+_{AW}(d)$ need not be regular. The following Lemma shows that $\tau^+_{AW}(d)$ on $CL(X)$ is always first countable.

Lemma 2.1. *Let (X,d) be a metric space. Then a local base for the topology $\tau^+_{AW}(d)$ at $A \in CL(X)$ consists of all sets of the form*

$$\Psi^+_A[n] = \{F \in CL(X) \colon F \cap S_d[x_0, n] \subset S_d[A, \tfrac{1}{n}]\},$$

where x_0 is arbitrary but fixed point and $n \in \mathbb{Z}^+$.

Proof. Let $n \in \mathbb{Z}^+$. We show that there is $B \in CLB(X)$ and $\varepsilon > 0$ such that

$$\Theta^+_A[B,\varepsilon] \subset \Psi^+_A[n].$$

Put $B = S_d[x_0, n]$ and $\varepsilon = \frac{1}{n}$. Then

$$\Theta^+_A[B,\varepsilon] \subset \Psi^+_A[n].$$

Now let $B \in CLB(X)$ and $\varepsilon > 0$. We show that there is $n \in \mathbb{Z}^+$ such that

$$\Psi^+_A[n] \subset \Theta^+_A[B,\varepsilon].$$

Let $m \in \mathbb{Z}^+$ be such that

$$B \subset S_d[x_0, m], \quad \frac{1}{m} < \varepsilon \text{ and } S_d[x_0, m] \cap A \neq \emptyset.$$

Put $n = 3m$. We claim that

$$\Psi^+_A[n] \subset \Theta^+_A[B,\varepsilon].$$

Let $F \in \Psi^+_A[n]$. Let $x \in B$. We wish to show that $d(x,A) - \varepsilon < d(x,F)$. There is $f \in F$ such that

$$d(x,F) + \frac{\varepsilon}{2} > d(x,f).$$

If $f \notin S_d[x_0, n]$ then

$$d(x,f) > 2m \qquad \text{and} \qquad d(x,A) < 2m,$$

thus

$$d(x,A) < 2m < d(x,f) < d(x,F) + \varepsilon,$$

i.e. $d(x,A) - \varepsilon < d(x,F)$.

If $f \in S_d[x_0, n]$ then there is $a \in A$ such that $d(a,f) < \frac{1}{n} = \frac{1}{3m} < \frac{\varepsilon}{3}$, Thus

$$d(x,A) \le d(x,a) \le d(x,f) + d(f,a) < d(x,F) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(x,F) + \varepsilon.$$

□

Remark 2.2. We finish this part by easy observation from Theorem A and Lemma 2.1 that $\tau^+_{AW}(d) \subset \sigma_d$, so we have

$$\tau^+_{AW}(d) \subset \sigma_d \subset \tau_{AW}(d) \quad \text{on} \quad CL(X).$$

3. The bounded proximal topology on function spaces

In this part we complete our results contained in [Ho2] concerning the relations between classical topologies and hypertopologies on function spaces

Let (X,d) and (Y,e) be metric spaces and let ϱ denote the box metric on $X \times Y$, i.e.

$$\varrho\big[(x_1,y_1),(x_2,y_2)\big] = \max\{d(x_1,y_1), e(y_1,y_2)\}.$$

If $f\colon X \to Y$ is a function, denote

$$G(f) = \big\{(x,f(x)) : x \in X\big\}$$

the graph of f. Denote $C(X,Y)$ the family of all continuous functions from X to Y. We can identify the members of $C(X,Y)$ with their graphs and consider $C(X,Y)$ as a subspace of $CL(X \times Y)$ with the induced mentioned topologies.

Denote:

$\tau_{AW}(\varrho)$ the Attouch–Wets topology on $C(X,Y)$,
σ_ϱ the bounded proximal topology on $C(X,Y)$,
$\tau^+_{AW}(\varrho)$ the upper Attouch–Wets topology on $C(X,Y)$,
τ_{CO} the compact-open topology and
τ_P the topology of pointwise convergence on $C(X,Y)$.

Notice at the beginning of this part that $\tau^+_{AW}(\varrho)$ on $C(X,Y)$ is T_1.

PROPOSITION 3.1. *Let (X,d) and (Y,e) be metric spaces. Then $\tau^+_{AW}(\varrho)$ on $C(X,Y)$ is T_1.*

Proof. Let $f \in C(X,Y)$. We show that $\{f\}$ is $\tau^+_{AW}(\varrho)$- closed. Let $g \in C(X,Y)$ such that $g \neq f$. There is $x \in X$ such that $f(x) \neq g(x)$. Let $\varepsilon > 0$ ($2\varepsilon < 1$) be such that

$$S_e\,[f(x), 2\varepsilon] \cap S_e\,[g(x), 2\varepsilon] = \emptyset.$$

The continuity of g at x implies that there is $\delta > 0$ such that

$$g(S_d\,[x,\delta]) \subset S_e\,[g(x),\varepsilon]\,.$$

We fix (x_0,y_0) in $X \times Y$ to serve as a center for ϱ-balls in $X \times Y$. There is $M \in \mathbb{Z}^+$ such that

$$(x,f(x)) \in S_\varrho\,[(x_0,y_0),M]$$

and $M > \max\{\frac{1}{\delta}, \frac{1}{\varepsilon}\}$. Then $\Psi^+_{G(g)}[M]$ does not contain $G(f)$. Suppose

$$G(f) \in \Psi^+_{G(g)}[M] .$$

Then

$$(x, f(x)) \in G(f) \cap S_\varrho[(x_0, y_0), M] \subset S_\varrho\left[G(g), \tfrac{1}{M}\right] ,$$

i.e. there is $(z, g(z))$ with $d(x,z) < \delta$ and $e(f(x), g(z)) < \varepsilon$. That is a contradiction. □

It is very easy to see that if (Y, e) is not bounded then $\tau^+_{AW}(\varrho)$ on $C(X,Y)$ is not Hausdorff.

Remark 3.2. The inclusions $\tau^+_{AW}(\varrho) \subset \sigma_\varrho \subset \tau_{AW}(\varrho)$ on $C(X,Y)$ are clear from Remark 2.2. From Lemma 2.1 and Theorem A it is very easy to see that

$$\sigma_\varrho \subset \tau^+_{AW}(\varrho) \vee \tau_p ,$$

where $\tau^+_{AW}(\varrho) \vee \tau_p$ is a topology which is the least upper bound of $\{\tau^+_{AW}(\varrho), \tau_p\}$.

If Y is a bounded metric space, then from Remark 2.4 in [Ho2] we can see that also $\tau^+_{AW}(\varrho) \vee \tau_p \subset \sigma_\varrho$, so if Y is a bounded metric space then we have

$$\tau^+_{AW}(\varrho) \vee \tau_p = \sigma_\varrho .$$

From Theorem 3.5 in [Ho2] we can deduce that if X is a locally connected metric space then $\sigma_\varrho = \tau^+_{AW}(\varrho) \vee \tau_p$ also, since under this condition on X σ_ϱ is finer than τ_{CO} on $C(X,Y)$. Example 1 in [Ho1] shows that if X is not locally connected then τ_p need not be weaker than σ_ϱ on $C(X,Y)$. This example also shows that if Y is not a bounded metric space then σ_ϱ need not be finer than τ_p on $C(X,Y)$.

We finish this part by two following characterizations:

THEOREM 3.3. *Let (X, d) be a dense in itself metric space. The following are equivalent:*

(1) *Every bounded subset of X is totally bounded;*
(2) *For every metric space (Y, e) in which bounded subsets are totally bounded $\tau_{AW}(\varrho) = \sigma_\varrho$ on $C(X,Y)$ where ϱ is the box metric on $X \times Y$.*

Proof. (1) $\implies$ (2) Let (Y, e) be a metric space in which every bounded set is totally bounded. It is easy to see that every ϱ- bounded set in $X \times Y$ (ϱ

is the box metric on $X \times Y$) is ϱ- totally bounded. Thus by the result in [BL2] $\tau_{AW}(\varrho) = \sigma_\varrho$ on $CL(X \times Y)$, i.e. $\tau_{AW}(\varrho) = \sigma_\varrho$ on $C(X,Y)$.

(2) $\Longrightarrow$ (1) Suppose there is a bounded set A in (X,d) which is not totally bounded. Put $Y = [0,1]$ with the usual metric which we denote by e. We show that $\tau_{AW}(\varrho)$ is not weaker than σ_ϱ on $C(X,Y)$ where ϱ is the box metric on $X \times Y$. We fix $(x_0,y_0) \in X \times Y$ to serve as a center for ϱ-balls in $X \times Y$.

There is an infinite subset $\{x_1, \dots x_n, \dots\}$ of A and ε ($0 < \varepsilon < 1$) such that $d(x_i,x_j) > 2\varepsilon$ for all $x_i \neq x_j$. For every $n \in \mathbb{Z}^+$ choose $y_n \in X \setminus \{x_n\}$ such that $d(x_n,y_n) < \frac{\varepsilon}{n}$ and put $\eta_n = d(x_n,y_n)$. Let $k \in \mathbb{Z}^+$. Define the function $f_k : X \to Y$ as follows:

$$f_k(z) = \begin{cases} 1 - (\frac{d(z,x_n)}{\eta_n}) & \text{if for some } n \leq k \quad z \in S_d[x_n,\eta_n] \\ 0 & \text{otherwise,} \end{cases}$$

and $f : X \to Y$ by

$$f(z) = \begin{cases} 1 - (\frac{d(z,x_n)}{\eta_n}) & \text{if for some } n \quad z \in S_d[x_n,\eta_n] \\ 0 & \text{otherwise.} \end{cases}$$

Clearly for each $k \in \mathbb{Z}^+$ the function f_k is continuous. Since the sequence $\{x_n\}$ has no cluster point the function f is also continuous.

It is very easy to see that $\{f_n\}$ σ_ϱ-converges to f. But $\{f_n\}$ does not $\tau_{AW}(\varrho)$-converge to f. (Let $n_0 \in \mathbb{Z}^+$ be such that $A \subset S_d[x_0,n_0]$ and $\frac{1}{n_0} < \varepsilon$. Then for each $n \in \mathbb{Z}^+$ $G(f_n) \notin \Sigma_{n_0}[G(f)]$.) □

THEOREM 3.4. *Let (Y,e) be a metric space. The following are equivalent:*

(1) *Every bounded subset of Y is totally bounded;*
(2) *For every metric space (X,d) in which bounded subsets are totally bounded $\tau_{AW}(\varrho) = \sigma_\varrho$ on $C(X,Y)$ where ϱ is the box metric on $X \times Y$.*

Proof. (1) $\Longrightarrow$ (2) Let (X,d) be a metric space in which every bounded set is totally bounded. It is easy to see that every ϱ-bounded set in $X \times Y$ is ϱ-totally bounded. Thus by the result in [BL2] $\tau_{AW}(\varrho) = \sigma_\varrho$ on $CL(X \times Y)$, i.e. $\tau_{AW}(\varrho) = \sigma_\varrho$ on $C(X,Y)$.

(2) $\Longrightarrow$ (1) Suppose there is a bounded set B in (Y,e) which is not totally bounded. Put $X = \{1, \frac{1}{2}, \dots \frac{1}{n}, \dots\}$ with the usual metric d. We show that $\tau_{AW}(\varrho)$ is not weaker than σ_ϱ on $C(X,Y)$. There is an infinite subset E of B $\{y_1, y_2, \dots y_n, \dots\}$ and $\varepsilon > 0$ such that $e(y_i,y_j) > 2\varepsilon$ for all $y_i \neq y_j$. Let f be the bijection from $X \to E$.

For each $n \in \mathbb{Z}^+$ put

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad x > \frac{1}{n} \\ f(\frac{1}{n}) & \text{if} \quad x \le \frac{1}{n}. \end{cases}$$

Since X is a discrete space, the functions $f,\ f_1,\ f_2,\ \dots\ f_n, \dots$ are continuous. It is very easy to verify that $\{f_n\}$ σ_ϱ-converges to f but $\{f_n\}$ does not $\tau_{AW}(\varrho)$ converge to f. □

4. The bounded proximal topology on continuous dual

Now we look at the weakenings of the Attouch–Wets topology applied on the space of continuous linear functionals. One of the most interesting properties of the Attouch–Wets topology applied on the space of continuous linear functionals is its agreement with the norm topology. Other results in this field can be found in [Be2], [Ho1].

It is the purpose of this section to show the relations between the strong topology and the bounded proximal topology on the space of continuous linear functionals.

In the sequel, X will be a topological linear space metrizable with an invariant metric d, with the origin Θ. X^* will be a space of all continuous linear functions from X to $\mathbb{R}$. We will consider the usual metric on $\mathbb{R}$. In what follows, the product $X \times \mathbb{R}$ will be understood to be equipped with the box metric, denoted by ϱ.

We will consider the topologies $\tau^+_{AW}(\varrho)$, σ_ϱ and $\tau_{AW}(\varrho)$ on X^*. We fix the point $(\Theta, 0)$ to serve as a center for ϱ-balls in the mentioned topologies.

Let φ denote the strong topology on X^*, i.e. if

$$U_{\varepsilon,A} = \{f \in X^* :\ |f(x)| < \varepsilon \quad \text{for every } x \in A\},$$

then φ is generated by the family $\{U_{\varepsilon,A}\}$, where ε runs over all positive reals and A runs over all linearly bounded sets in X. (A is a linearly bounded set if for every neighbourhood V of Θ there is $n > 0$ such that $\lambda V \supset A$ for every λ: $|\lambda| > n$).

The following result seems to be surprising in comparison with [Ho1] where we required more and proved less.

THEOREM 4.1. *Let X be a topological linear space metrizable with an invariant metric d. Then the strong topology φ on X^* is weaker than $\tau^+_{AW}(\varrho)$ topology, where ϱ is the box metric on $X \times \mathbb{R}$.*

Proof. It is sufficient to prove that $\tau_{AW}^{+}(\varrho)$-convergence of any sequence $\{f_n\}$ from X^* to $f \in X^*$ implies φ-convergence of $\{f_n\}$ to f since $\tau_{AW}^{+}(\varrho)$ is first countable (Lemma 2.1).

Let $\{f_n\}$ be a sequence from X^* $\tau_{AW}^{+}(\varrho)$-convergent to a function $f \in X^*$. First we show that there is a neighbourhood O of Θ and $N_0 \in \mathbb{Z}^+$ such that for every $n \geq N_0$ and every $z \in O$ we have $|f_n(z)| \leq 1$. Suppose it is not true. There is a sequence $\{x_n\}$ d- convergent to Θ and a subsequence $\{h_n\}$ of $\{f_n\}$ such that $|h_n(x_n)| > 1$.

The continuity of f at Θ implies that there is δ $(0 < \delta < 1)$ such that

$$(1) \quad |f(z)| < \frac{1}{4} \qquad \text{for every } z \in S_d\,[\Theta, \delta]\ .$$

There is a balanced neighbourhood K of Θ such that $K \subset S_d\left[\Theta, \frac{\delta}{2}\right]$. Let $\eta > 0$ be such that $S_d\,[\Theta, \eta] \subset K$. Let $M \in \mathbb{Z}^+$ be such that $M \geq \max\{\frac{1}{\eta}, 4\}$. There is $N_1 \in \mathbb{Z}^+$ such that for every $n \geq N_1$

$$G(h_n) \in \Psi^{+}_{G(f)}\,[M]$$

and $x_n \in K$. Let $n \geq N_1$. The point $y_n = \frac{x_n}{h_n(x_n)} \in K$ and $h_n(y_n) = 1$. Thus there is a point $(z, f(z))$ such that

$$\varrho\,[(z, f(z)), (y_n, h_n(y_n))] < \frac{1}{M}\,,$$

i.e. $d(z, \Theta) < \delta$ and $|f(z) - 1| < \frac{1}{4}$ and that is a contradiction to (1).

Thus there is $\alpha > 0$ $(\alpha < 1)$ such that

$$|f(z)| \leq 1 \quad \text{and} \quad |f_n(z)| \leq 1$$

for every $n \in \mathbb{Z}^+$ and $z \in S_d\,[\Theta, \alpha]$. We show that $\{f_n\}$ converges uniformly to f on $S_d\,[\Theta, \alpha]$. Let $\varepsilon > 0$ $(\varepsilon < 1)$. The uniform continuity of f implies that there is $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

whenever $d(x, y) < \delta$.

Put $\eta = \min\{\frac{\varepsilon}{2}, \delta\}$. Let $k \in \mathbb{Z}^+$ be such that $k > \frac{1}{\eta}$. There is $N_0 \in \mathbb{Z}^+$ such that for every $n \geq N_0$

$$G(f_n) \in \Psi^{+}_{G(f)}\,[k]\ .$$

Then for every $n \geq N_0$ and every $x \in S_d[\Theta, \alpha]$

$$|f_n(x) - f(x)| < \varepsilon .$$

Let $n \geq N_0$ and $x \in S_d[\Theta, \alpha]$. Then

$$\big(x, f_n(x)\big) \in S_\varrho[(\Theta, 0), k] ,$$

i.e. there is $z \in X$ such that

$$\varrho\big[\big(x, f_n(x)\big), \big(z, f(z)\big)\big] < \frac{1}{k} ,$$

i.e. $d(x, z) < \frac{1}{k} < \delta$ and

$$|f_n(x) - f(z)| < \frac{1}{k} < \frac{\varepsilon}{2} .$$

Since $|f(z) - f(x)| < \frac{\varepsilon}{2}$ we have $|f_n(x) - f(x)| < \varepsilon$.

From this observation φ-convergence of $\{f_n\}$ to f is obvious. □

R e m a r k 4.2. If X is a topological linear space metrizable with an invariant metric d, we have also the inclusions $\varphi \subset \sigma_\varrho$ on X^* and $\tau_{AW}(\varrho) \supset \varphi$. If X is normable then by [Be4] $\varphi = \tau_{AW}(\varrho)$ on X^* and thus

$$\varphi = \tau_{AW}^{+}(\varrho) = \sigma_\varrho = \tau_{AW}(\varrho)$$

on X^*.

PROPOSITION 4.3. *Let X be a topological linear space metrizable with an invariant metric d. Then on X^* $\tau_{AW}^{+}(\varrho) = \sigma_\varrho$.*

P r o o f . The inclusion $\tau_{AW}^{+}(\varrho) \subset \sigma_\varrho$ is clear from definitions. We prove that $\sigma_\varrho \subset \tau_{AW}^{+}(\varrho)$. Again it is sufficient to prove that $\tau_{AW}^{+}(\varrho)$-convergence of any sequence $\{f_n\}$ from X^* to $f \in X^*$ implies σ_ϱ-convergence of $\{f_n\}$ to f for $\tau_{AW}^{+}(\varrho)$ is first countable. Let $\{f_n\}$ be a sequence from X^* $\tau_{AW}^{+}(\varrho)$-convergent to a function $f \in X^*$. By Theorem 4.1 the sequence $\{f_n\}$ is pointwise convergent to f. It is very easy to see from Remark 3.2 that $\{f_n\}$ σ_ϱ-converges to f. □

In the following part we introduce the relation between the convergence of the graphs of continuous linear functionals and the convergence of their level sets.

For each nonzero $f \in X^*$ and each $\alpha \in \mathbb{R}$ the level set $f^{-1}(\alpha)$ is a closed hyperplane in X. We will see that σ_d ($\tau_{AW}(d)$) convergence of level sets of nonzero continuous linear functionals at fixed heights corresponds to σ_ϱ ($\tau_{AW}(\varrho)$ resp.) convergence for the functionals themselves.

THEOREM 4.4. *Let X be a topological linear space metrizable with an invariant metric d. Let $f,\ f_1,\ f_2,\dots$ be nonzero element from X^*. The following are equivalent:*

(1) *For each $\alpha \in \mathbb{R}$ $\{f_n^{-1}(\alpha)\}$ σ_d-converges to $f^{-1}(\alpha)$;*
(2) *$\{f_n\}$ σ_ϱ-converges to f.*

Proof. (1) $\Longrightarrow$ (2) Assume that there is $M \in \mathbb{Z}^+$ and there is a subsequence $\{h_n\}$ of $\{f_n\}$ such that

$$G(h_n) \notin \Psi^+_{G(f)}[M]\,,$$

i.e. there is a sequence $\{(x_n, h_n(x_n))\}$ such that

$$(x_n, h_n(x_n)) \in S_\varrho[(\Theta,0),M] \setminus S_\varrho\left[G(f),\frac{1}{M}\right]$$

for each n. Without loss of generality we can suppose that $\{h_n(x_n)\}$ converges to some $\alpha \in \mathbb{R}$.

Let $\delta < \frac{1}{2M}$. There is a balanced neighbourhood U of Θ such that

$$U \subset S_d[\Theta,\delta] \quad \text{and} \quad |f(z)| < \frac{1}{2M}$$

for all $z \in U$. Let U_1 be a balanced neighbourhood of Θ such that $U_1+U_1 \subset U$ and let δ_1 be such that $S_d[\Theta,\delta_1] \subset U_1$. Let $z_0 \in U_1$ be such that $f(z_0) > 0$ and put $B = f(z_0)$. Then f takes on U_1 all values between $-B$ and B. σ_d-convergence of $\{h_n^{-1}(B)\}$ to $f^{-1}(B)$ implies that there is $K \in \mathbb{Z}^+$ such that for each $k \geq K$

$$h_k^{-1}(B) \in \Phi_{f^{-1}(B)}[N, z_0]\,, \tag{$*$}$$

where $N \in \mathbb{Z}^+$ be such that $N > \max\{\frac{1}{\delta_1}, \frac{2}{B}\}$.

($*$) implies that for every $k \geq K$ h_k takes on U all values from $[-B,B]$. (Let $k \geq K$. From ($*$) we have

$$d\big(z_0, h_k^{-1}(B)\big) < \frac{1}{N} < \delta_1\,,$$

i.e. there is $u \in h_k^{-1}(B)$ such that $d(z_0,u) < \delta_1$, i.e. $u \in U$ and $h_k(u) = B$.)

There is $N_1 \in \mathbb{Z}^+$ such that for every $n \geq N_1$

$$|h_n(x_n) - \alpha| < \frac{1}{N} < \frac{B}{2}\,.$$

For every $k \geq \max\{N_1, K\}$ there is $z_k \in X$ such that

$$d(z_k, x_k) < \frac{1}{2M} \qquad \text{and} \qquad h_k(z_k) = \alpha\,.$$

There is $K_1 \in \mathbb{Z}^+$ such that for every $k \geq K_1$

$$h_k^{-1}(\alpha) \in \Psi^+_{f^{-1}(\alpha)}[N]\,.$$

Put $H = \max\{K, K_1, N_1\}$. Let $k \geq H$. Then $z_k \in h_k^{-1}(\alpha)$, i.e. there is $u \in f^{-1}(\alpha)$ such that

$$d(z_k, u) < \frac{1}{N} < \delta_1 < \frac{1}{2M}.$$
$$|h_k(x_k) - f(u)| = |h_k(x_k) - \alpha| < \frac{1}{N} < \frac{1}{2M}$$

and

$$d(x_k, u) \leq d(z_k, u) + d(x_k, z_k) < \frac{1}{M}$$

and that is a contradiction.

To finish the proof of implication (1) $\Rightarrow$ (2) it is sufficient to verify that for every $x \in X$

$$\{\varrho((x, f(x)), G(f_n))\}$$

converges to 0.

Let $x_0 \in X$. Let $N \in \mathbb{Z}^+$. The convergence of

$$\{d(x_0, f_n^{-1}(f(x_0)))\} \quad \text{to} \quad 0$$

implies that there is $K \in \mathbb{Z}^+$ such that

$$d(x_0, f_k^{-1}(f(x_0))) < \frac{1}{N}$$

for every $k \geq K$. Let $k \geq K$ and

$$u_k \in f_k^{-1}(f(x_0)) \quad \text{such that } d(x_0, u_k) < \frac{1}{N}\,.$$

Then

$$\varrho\big((x_0, f(x_0)), G(f_k)\big) \le d(u_k, x_0) < \frac{1}{N}.$$

(2) $\implies$ (1) Let $\alpha \in \mathbb{R}$. Let $m \in \mathbb{Z}^+$. First we show that there is $K \in \mathbb{Z}^+$ such that for every $k \ge K$

$$f_k^{-1}(\alpha) \in \Psi^+_{f^{-1}(\alpha)}[m].$$

There is a balanced neighbourhood O of Θ such that

$$O \subset S_d\left[\Theta, \tfrac{1}{2m}\right] \quad \text{and} \quad |f(z)| < \tfrac{1}{n}$$

for each $z \in O$, where $n > \max\{m, |\alpha|\}$. Let $z \in O$ be such that $f(z) > 0$ and put $B = f(z)$. Then f takes on O all values from the interval $[-B, B]$. Let $N \in \mathbb{Z}^+$ be such that $N > \frac{2}{B}$. There is $K \in \mathbb{Z}^+$ such that for every $k \ge K$

$$G(f_k) \in \Psi^+_{G(f)}[N].$$

We show that for each $k \ge K$

$$f_k^{-1}(\alpha) \in \Psi^+_{f^{-1}(\alpha)}[m].$$

Let $k \ge K$. Let

$$x \in f_k^{-1}(\alpha) \cap S_d[\Theta, m].$$

Then

$$\big(x, f_k(x)\big) \in S_\varrho[(\Theta, 0), N],$$

i.e. there is $(z, f(z))$ such that

$$d(x, z) < \frac{1}{N} < \frac{1}{2m} \quad \text{and} \quad |\alpha - f(z)| < \frac{B}{2}.$$

There is $u \in S_d\left[\Theta, \frac{1}{2m}\right]$ such that $f(u) = \alpha - f(z)$, i.e. $\alpha = f(z + u)$ and $d(u + z, x) \le \frac{1}{m}$.

Secondly we show that for each $x \in f^{-1}(\alpha)$ there is $K \in \mathbb{Z}^+$ such that for every $k \ge K$

$$d(x, f_k^{-1}(\alpha)) < \frac{1}{m}.$$

Let $x \in f^{-1}(\alpha)$. There is a balanced neighbourhood O of Θ such that

$$O \subset S_d\left[\Theta, \frac{1}{2m}\right] \quad \text{and} \quad |f(z)| < \frac{1}{2m}$$

for every $z \in O$. Let O_1 be a balanced neighbourhood of Θ such that $O_1+O_1 \subset O$. Let $\delta > 0$ be such that $S_d[\Theta, \delta] \subset O_1$. Let $z_0 \in O_1$ be such that $f(z_0) > 0$ and put $B = f(z_0)$. Then f takes on O_1 all values from $[-B, B]$. Let $M \in \mathbb{Z}^+$ be such that $M > \max\{\frac{1}{\delta}, \frac{2}{B}\}$. There is $K \in \mathbb{Z}^+$ such that for every $k \geq K$

$$\varrho\big((z_0, f(z_0)), G(f_k)\big) < \frac{1}{M} \quad \text{and} \quad \varrho\big((x, \alpha), G(f_k)\big) < \frac{1}{M}.$$

From the first inequality we obtain that for every $k \geq K$ f_k takes on O all values from $\left[\frac{-B}{2}, \frac{B}{2}\right]$. Let $k \geq K$. There is $y_k \in X$ such that

$$\varrho\big((x, \alpha), (y_k, f_k(y_k)\big) < \frac{1}{M},$$

i.e.

$$d(x, y_k) < \delta \quad \text{and} \quad |\alpha - f_k(y_k)| < \frac{B}{2}.$$

Thus there is

$$u_k \in S_d\left[\Theta, \frac{1}{2m}\right] \quad \text{such that } f_k(u_k) = \alpha - f_k(y_k),$$

i.e.

$$\alpha = f_k(u_k + y_k) \quad \text{and} \quad d(x, u_k + y_k) < \frac{1}{m}.$$

Thus for each $k \geq K$ $d\big(x, f_k^{-1}(\alpha)\big) < \frac{1}{m}$. □

COROLLARY 4.5. *Let X be a topological linear space metrizable with an invariant metric d. Let $f, f_1, f_2, \dots$ be nonzero elements from X^*. If for every $\alpha \in \mathbb{R}$ $f_n^{-1}(\alpha)$ σ_d-converges to $f^{-1}(\alpha)$ then $\{f_n\}$ converges to f in the strong topology φ.*

The following fact can be proved similarly as Theorem 4.4.

THEOREM 4.6. *Let X be a topological linear space metrizable with an invariant metric d. Let $f, f_1, f_2, \dots$ be nonzero elements from X^*. The following are equivalent:*

(1) *For every $\alpha \in \mathbb{R}$ $\{f_n^{-1}(\alpha)\}$ $\tau_{AW}(d)$-converges to $f^{-1}(\alpha)$;*

(2) *$\{f_n\}$ $\tau_{AW}(\varrho)$-converges to f.*

The following theorem gives a characterization of normable linear spaces by using the agreement of the mentioned topologies.

THEOREM 4.7. *Let X be a locally convex topological linear space metrizable with an invariant metric d and ϱ be the box metric on $X \times \mathbb{R}$. The following are equivalent:*

(1) *X is normable;*
(2) *$\varphi = \tau_{AW}(\varrho)$ on X^*;*
(3) *$\varphi = \tau_{AW}^{+}(\varrho)$ on X^*;*
(4) *$\varphi = \sigma_{\varrho}$ on X^*;*
(5) *For every sequence $f, f_1, f_2, \dots$ of nonzero elements from X^* $\{f_n\}$ φ-converges to f if and only if for every $\alpha \in \mathbb{R}$, $\{f_n^{-1}(\alpha)\}$ σ_d-converges to $f^{-1}(\alpha)$.*

Proof. (1) $\Longrightarrow$ (2) The proof of this implication can be found in [Be4].

(2) $\Longrightarrow$ (3) It is clear from Remark 4.2.

(3) $\Longrightarrow$ (4) It is clear from Proposition 4.3.

(4) $\Longrightarrow$ (5) This implication is clear from Theorem 4.4.

(5) $\Longrightarrow$ (1) Suppose X is not normable. Put $A = S_d[\Theta, 1]$. Let V be an absolutely convex neighbourhood of Θ such that $\mathrm{cl}V \neq X$ and $V \subset A$. (By an absolutely convex set we mean a set which is at the same time convex and balanced.) Put $H = \frac{1}{2}V$. Clearly H cannot be linearly bounded, i.e. there is an absolutely convex neighbourhood U of Θ with the following property: for every $n \in \mathbb{Z}^+$ there is $k_n > n$ such that H is not a subset of $k_n U$. There is a sequence $\{x_n\} \subset H$ such that $x_n \notin k_n U$ for every $n \in \mathbb{Z}^+$, i.e. $\frac{x_n}{n} \notin U$ for every $n \in \mathbb{Z}^+$. Without loss of generality we can also suppose that $\frac{x_n}{n} \notin \mathrm{cl}U$ for each $n \in \mathbb{Z}^+$. For every $n \in \mathbb{Z}^+$ there is $h_n \in X^*$ [RR] such that

$$h_n\left(\frac{x_n}{n}\right) > 1 \quad \text{and} \quad |h_n(x)| \leq 1$$

for each $x \in U$. Put $g_n = \frac{h_n}{n}$. Then $\{g_n\}$ φ-converges to zero function.

There is a nonzero function $L \in X^*$ such that $|L(x)| \leq \frac{1}{4}$ on V. Then the sequence $\{(g_n + L)\}$ φ-converges to L. We show that $\{(g_n + L)^{-1}(1)\}$ does not σ_d-converge to $L^{-1}(1)$. It is easy to verify that for each $n \in \mathbb{Z}^+$ there is a point $y_n \in H$ such that $g_n(y_n) = 1$. Thus for each $n \in \mathbb{Z}^+$ the following inequalities hold

$$\frac{3}{4} < (g_n + L)(y_n) < \frac{5}{4}.$$

If

$$(g_n + L)(y_n) > 1 \quad \text{then} \quad z_n = \frac{y_n}{(g_n + L)(y_n)} \in H$$

and $(g_n + L)(z_n) = 1$. If

$$(g_n + L)(y_n) < 1 \quad \text{then} \quad z_n = \frac{y_n}{(g_n + L)(y_n)} \in \frac{4}{3}H = \frac{4}{3}\frac{1}{2}V = \frac{2}{3}V$$

and $(g_n + L)(z_n) = 1$.

Let $k \in \mathbb{Z}^+$ be such that $S_d\left[\Theta, \frac{1}{k}\right] \subset \frac{1}{3}V$. Then for each $n \in \mathbb{Z}^+$

$$(g_n + L)^{-1}(1) \notin \Psi^+_{L^{-1}(1)}[k].$$

(Let $n \in \mathbb{Z}^+$. Then $z_n \in \frac{2}{3}V$. Suppose there is $u \in L^{-1}(1)$ such that $d(z_n, u) < \frac{1}{k}$ i.e. $u - z_n \in \frac{1}{3}V$ and thus $u \in V$. That is a contradiction since $|L(x)| \le \frac{1}{4}$ for every $x \in V$.) □

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Received November 4, 1992

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