Tatra Mountains Math. Publ., 1 (1992), 83–87

TATRA MOUNTAINS
Mathematical Publications

D—POSETS OF FUZZY SETS

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1. Introduction

The Klement axiomatic definition of probability of fuzzy sets ([1]) is based on the fuzzy measurable space and on the fuzzy probability measure, which replace the measurable space and the probability measure of the classical Kolmogorov model. Fuzzy measurable space is a system of functions (of fuzzy sets), $F \subseteq [0,1]^X$, closed with respect to the constant functions, the difference $1-f$, and the supremums of arbitrary sequences from F.

The fundamental notions of the quantum theory that is built on the principle of fuzzy sets ([4]) are F-quantum space, F-observable, F-state, which are analogies of the quantum logic, observable and the state in the quantum logic theory ([3]).

In both models we consider the Zadeh operations of the union and intersection of fuzzy sets. If the complement of fuzzy set f is the fuzzy set $f^\perp = 1-f$, and if $g \subseteq f$, we can define the difference of fuzzy sets as follows:

$$f \setminus g := f \wedge g^\perp = \min\{f, g^\perp\}.$$

However, in this case the difference has not such properties as in the case of crisp sets. For example, $f \setminus (f \setminus g) \neq g$, in general. For example, if $f = 0.6$; $g = 0.2$; then $f \setminus (f \setminus g) = 0.4 \neq g$.

2. D–posets of fuzzy sets

Now we shall define the difference of fuzzy sets such that its properties will be equal to the difference of crisp sets. Finally, we shall define the structure of fuzzy sets, such that we can obtain the probability results.

DEFINITION 1. Let $F \subseteq [0,1]^X$ be a system of fuzzy subsets of a non-empty set X. We shall say that on the system F is defined the *difference of fuzzy sets*, if for every f,g from F, $g \subseteq f$ ($g(t) \le f(t)$ for any $t \in X$) there exists a fuzzy set $f \setminus g$ from F fulfilling the following conditions:

(i) $f \setminus g \subseteq f$;
(ii) $f \setminus (f \setminus g) = g$;
(iii) if f,g,h are the fuzzy sets of F, $h \subseteq g \subseteq f$, then $f \setminus g \subseteq f \setminus h$ and $(f \setminus h) \setminus (f \setminus g) = g \setminus h$.

The system F with the difference will be called *D-poset* of fuzzy sets.

We note, that the fuzzy set $f \setminus g$ in Definition 1 is defined uniquely.

EXAMPLES. Let $F = [0,1]^X$, $f,g \in F, f \subseteq g$. Then the following operations are the differences of fuzzy sets:

1. $f \setminus g := f - g$ i.e. $(f \setminus g)(t) = f(t) - g(t)$, $t \in X$.
2. $f \setminus g := \sqrt{f^2 - g^2}$, i.e. $(f \setminus g)(t) = \sqrt{f^2(t) - g^2(t)}$, $t \in X$.
3. The generalization of the previous cases: $f \setminus g := u^{-1}(u(f) - u(g))$, i.e. $(f \setminus g)(t) = u^{-1}\Big(u\big(f(t)\big) - u\big(g(t)\big)\Big)$, $t \in X$,

where u is a continuous strictly increasing function on the unit interval $[0,1]$, such that $u(0) = 0$.

Indeed:

(i) $u(f) - u(g) \subseteq (f)$ and $u^{-1}(u(t) - u(g)) \subseteq u^{-1}(u(f)) = f$.
(ii) $f \setminus (f \setminus g) = u^{-1}\left[u(f) - u\left(u^{-1}(u(f) - u(g))\right)\right] = u^{-1}[u(g)] = g$.
(iii) if $h \subseteq g \subseteq f$, then $f \setminus g = u^{-1}(u(f) - u(g))$ and $f \setminus h = u^{-1}(u(f) - u(h))$, and so $f \setminus g \subseteq f \setminus h$.

Finally, $(f \setminus h) \setminus (f \setminus g) = u^{-1}(u(f) - u(h) - u(f) + u(g)) = u^{-1}(u(g) - u(h)) = g \setminus h$. Q.E.D.

It is not difficult to prove the following proposition.

PROPOSITION 1. *Let F be a D-poset of fuzzy sets and $f,g,h,u \in F$. Then the following assertions are true.*

(1) *If $f \subseteq g \subseteq h$ then $g \subseteq f \subseteq h \setminus f$ and $(h \setminus f) \setminus (g \setminus f) = h \setminus g$.*
(2) *If $g \subseteq h$ and $f \subseteq h \setminus g$ then $g \subseteq h \setminus f$ and $(h \setminus g) \setminus f = (h \setminus f) \setminus g$.*
(3) *If $f \subseteq g \subseteq h$ then $f \subseteq h \setminus (g \setminus f)$ and $(h \setminus (g \setminus f)) \setminus f = h \setminus g$.*
(4) *If $f \subseteq h$ and $g \subseteq h$ then $h \setminus f = h \setminus g$ if and only if $f = g$.*
(5) *If $u \in P$, $u \subseteq f \subseteq h$, $u \subseteq g \subseteq h$ then $h \setminus f = g \setminus u$ if and only if $h \setminus g = f \setminus u$.*

In the following, with respect to the probability theory, we shall consider such a D-poset F of fuzzy sets that the following conditions are fulfilled:

(iv) if $(f_n)_{n \in \mathbb{N}}$ is a sequence of F, $f_n \subseteq f_{n+1}$, $n \in \mathbb{N}$, then $\bigvee_{n \in \mathbb{N}} f_n \in F$ $\left((\bigvee_{n \in \mathbb{N}} f_n)(t) = \sup\{f_n(t), \quad t \in X\}\right)$;

(v) if $1_X(t) = 1$ for every $t \in X$, then $1_X \in F$.

DEFINITION 2. The system F with the axioms (i)–(v) will be called *D-σ-poset* of fuzzy sets with the greatest element.

PROPOSITION 2. *Let F be a D-σ-poset of fuzzy sets with 1_X. Then the following assertions are true.*

(1) $1_X \setminus 1_X = 0_X$ *i.e.* $0_X(t) = 0$ *for every* $t \in X$;
(2) $f \setminus 0_X = f$ *for any* $f \in F$;
(3) $f \setminus f = 0_X$ *for any* $f \in F$;
(4) *if* $f, g \in F$, $g \subseteq f$, *then* $f \setminus g = 0_X$ *if and only if* $f = g$;
(5) *if* $f, g \in F$, $g \subseteq f$, *then* $f \setminus g = f$ *if and only if* $g = 0_X$.

3. Observables and states on d–Σ–posets of fuzzy sets

We note that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of the real line $\mathbb{R}$ with the usual difference of sets is D-σ-poset of crisp sets with the greatest element $\mathbb{R}$.

DEFINITION 3. Let F be a D-σ-poset of fuzzy sets with the greatest element 1_X. The ***probability measure*** on F is a map $m : F \to [0,1]$ such that:

(i) $m(1_X) = 1$;
(ii) if $(f_n)_{n\in\mathbb{N}}$ is a sequence of F such that f_n increases to the f, i.e., $\bigvee\limits_{n\in\mathbb{N}} f_n = f$, then

$$m(f) = m(f_1) + \sum_{n=2}^{\infty} m(f_n \setminus f_{n-1}).$$

We remark: 1) If D-σ-poset F is closed with respect to the Zadeh operations of the union and intersection of fuzzy sets and $f \setminus g := f - g$ for $g \subseteq f$, then the measure m is a valuation.

Indeed:

$$(f \vee g) \setminus f = \begin{Bmatrix} g - f & \text{if} & f \subseteq g \\ 0 & \text{if} & g \subseteq f \end{Bmatrix} = g \setminus (f \wedge g).$$

Then we have $m(f \vee g) - m(f) = m(g) - m(f \wedge g)$ and so

$$m(f \vee g) + m(f \wedge g) = m(f) + m(g).$$

2) If F is a D-σ-poset of fuzzy sets with 1_X, t_0 is an arbitrary point from X, then the mapping $m : F \to [0,1]$ such that $m(f) = f(t_0)$ is a probability measure on F.

The role of the random variable from the classic theory will be taken as an observable.

DEFINITION 4. Let F be a D-σ-poset of fuzzy sets with 1_X . Let $\mathcal{B}(\mathbb{R})$ be a Borel σ-algebra of the real line $\mathbb{R}$. The mapping $x : \mathcal{B}(\mathbb{R}) \to F$ is said to be *observable on* F if the following conditions are fulfilled:

(i) $x(\mathbb{R}) = 1_X$;

(ii) if $(A_n)_{n\in\mathbb{N}}$ is a sequence of Borel sets such that $A_n \subseteq A_{n+1}$, $n \in \mathbb{N}$, then $x(A_n) \subseteq (A_{n+1})$ and $x(\bigcup_{n\in\mathbb{N}} A_n) = \bigvee_{n\in\mathbb{N}} x(A_n)$.

(iii) if A, B are the Borel sets, $\subseteq B$, then $x(B \setminus A) = x(B) \setminus x(A)$.

We remark that: — the observable x is not a σ-homomorphism, in general

— the range of the observable x is not a Boolean σ-algebra, in general.

THEOREM 1. *Let F be a D-σ-poset of fuzzy sets with 1_X, x be an observable on F. Then the mapping $m_x : \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by the formula*

$$m_x(E) := m(x(E))$$

is a probability measure on $\mathcal{B}(\mathbb{R})$.

Proof. 1. $m_x(\mathbb{R}) = m(x(\mathbb{R})) = m(1_x)$.

2. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ is a sequence of mutually disjoint subsets. Put $B_n = \bigcup_{i=1}^{n}$. Then we have

$$\begin{aligned} m_x(\bigcup_{n\in\mathbb{N}} E_n) &= m\left(x\Big(\bigcup_{n\in\mathbb{N}} B_n\Big)\right) = m\left(\bigvee_{n\in\mathbb{N}} x(B_n)\right) = \\ &= m(x(B_1)) + \sum_{n=2}^{\infty} m(x(B_n) \setminus x(B_{n-1})) = \\ &= m(x(E_1)) + \sum_{n=2}^{\infty} m(x(E_n)) = \\ &= \sum_{n\in\mathbb{N}} m_x(E_n). \end{aligned}$$

□

Finally we shall define the mean value of the observable.

DEFINITION 5. Let x be an observable on D-σ-poset F of fuzzy sets, m be a probability measure on F. A *mean value* in the measure m is an integral

$$E(x) = \int_X t \, \mathrm{d}m_x,$$

if this integral exists and is finite.

We remark that if we shall consider any partially ordered set that is a D-poset as well ([2]), we obtain the generalization of the quantum logic theory ([3]).

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