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BOOLEAN METHODS IN F–QUANTUM SPACES

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ABSTRACT. In this paper, we present a Boolean, point-free characterization of fuzzy observables, using Boolean-valued Dedekind cuts and the theory of Boolean powers. In the second part of the paper we study the links of Quantum spaces with the theory of orthospaces and its associated tolerance spaces. Finally in the third part using a soft Boolean algebra, we construct a Boolean model which incorporates all the previous ideas.

Introduction

Starting from Suppes' (1966) model of quantum probability and replacing subsets with indicator functions, one can immediately see that the indicators can be generalized to:

(i) Functions with values in $[0,1]$, i.e., ordinary fuzzy sets or,

(ii) Functions with values in a Boolean algebra, getting in this way Boolean valued models and $\mathbb{B}$-fuzzy sets [5], [7].

Dvurečenskij and Riečan [11] [12] combining the Suppes approach with the Piasecki concept of soft fuzzy σ-algebra and fuzzy p-measures, suggested a generalization of Suppes' model, known as F-Quantum space. This model has been suggested in order to grasp and express the vagueness associated with quantum mechanical events. See [13], for a review of Fuzzy Quantum Spaces.

A corresponding calculus of fuzzy observables has been developed, and results such as: the Randon-Nikodým theorem, the central limit theorem, a generalized Loomis-Sikorski theorems, etc. have been obtained, see [18], [13], for a review of these developments.

The similarity of these results with the ones in classical probability, and the representations of fuzzy observables by ordinary random variables, led many

Key words: F-Quantum spaces, fuzzy observables, Boolean valued Dedekind cuts, Boolean powers, proximity or tolerance relation, orthogonality relation, orthospaces.

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researchers to look for methods of reduction of F-Quantum spaces to ordinary probability algebras. It is also true that this striking similarity exists between Piasecki's fuzzy probability spaces and a kind of point-free probability theory see [16].

In this paper we first give a representation of fuzzy observables using Boolean-valued Dedekind cuts, and then using some soft Boolean σ-algebra we construct a Boolean power model of the reals. The real numbers in this model are discrete fuzzy observables. We use this model of reals $\mathbb{R}[\mathbb{B}]$ as a non-Cantorian analogue of the Cantorian and absolute concept of a Hilbert space.

Next we introduce methods of reduction of this model, using ultrafilters and "tolerant ultrafilters" in order to construct a model for a quantum logic based on similarity and tolerance or proximity relations.

2. Soft probability σ-algebras

In this section we shall briefly review the various Boolean σ-algebras that can be derived from a soft probability space $(\Omega, \mathcal{F}, P)$, i.e. Ω is a non-empty set, called universum, $\mathcal{F}$ is a so called *soft fuzzy σ-algebra*, i.e., $\mathcal{F} \subseteq [0,1]^{\Omega}$ such that:

(i) $1_{\Omega}(\cdot) \in \mathcal{F}$.
(ii) $(\forall f \in \mathcal{F})[f \in \mathcal{F} \implies f^{\perp} \equiv 1 - f \in \mathcal{F}]$.
(iii) $f_i \in \mathcal{F}, \quad i \geq 1, \quad \implies \quad \cup_i f_i := \sup f_i \in \mathcal{F}$, where $\cup_i f_i := \sup f_i, \quad \cap_i f_i := \inf f_i$, and $a \Delta b := a \cap b^{\perp} \cup b \cap a^{\perp}$.
(iv) $\frac{1}{2} \notin \mathcal{F}$ where $\frac{1}{2}(\omega) := \frac{1}{2}$ for all $\omega \in \Omega$;

and $P(\cdot) : \mathcal{F} \to [0,1]$ is a *fuzzy P-measure*, i.e.

(i) $P(f \vee f^{\perp}) = 1, \qquad f \in \mathcal{F}$;

(ii) $P(\bigvee_{i=1}^{\infty} f_i) = \sum_{i=1}^{\infty} P(f_i)$, whenever $f_i \leq f_j^{\perp}, \quad i \neq j$.

Let

$$W_0(\mathcal{F}) : = \{a \wedge a' : a \in \mathcal{F}\},$$

the set of all W-empty sets, and

$$W_1(\mathcal{F}) : = \{a \vee a' : a \in \mathcal{F}\},$$

the set of all W-universums.

Let also, for each $a \in \mathcal{F}$,

$$H(a) := a^{-1}\left[(\tfrac{1}{2},1]\right] \qquad \text{(high values)},$$
$$M(a) := a^{-1}\left[\{\tfrac{1}{2}\}\right] \qquad \text{(middle values)},$$
$$L(a) := a^{-1}\left[[0,\tfrac{1}{2})\right] \qquad \text{(low values)},$$

then $\Omega = H(a) \cup M(a) \cup L(a),\ a \in \mathcal{F}$.

2.1 DEFINITION. A *σ-ideal on* $\mathcal{F}$ is a subset $I \subseteq \mathcal{F}$ such that, for all $a, b, a_i \in \mathcal{F},\ i \in N$, we have:

(i) $W_0 \subseteq I$,

(ii) $a \leq b\ \&\ b \in I \implies a \in I$,

(ii) $a_i \in I, i \in N \implies \bigvee_{i \in I} a_i \in I$,

(iv) $b \in W_1\ \&\ a \wedge b \in I \implies a \in I$.

From [11] we have the following:

2.2 THEOREM. *If $\mathbb{B}$ is a Boolean σ-algebra and $h : \mathcal{F} \to \mathbb{B}$ is a σ-homomorphism, then the kernel* $\ker(h) := \{a \in \mathcal{F} : h(a) = 0\}$ *is a σ-ideal on $\mathcal{F}$, conversely if I is a σ-ideal on $\mathcal{F}$, then the canonical map $h_I : \mathcal{F} \to \mathcal{F}/I$, where $h_I(a) := \{b \in \mathcal{F} : a \Delta b \in I\}$ is a σ-homomorphism and $\mathcal{F}/I$ is a Boolean σ-algebra.*

Some σ-ideals have been proposed so far:

(i) W_0 is not quite a σ-ideal since, although it belongs to the kernel of every σ-homomorphism, it is not the kernel of a σ-homomorphism. However

$$I_0 := \{a \in \mathcal{F} : (\exists c \in W_1)[a \wedge c \leq \tfrac{1}{2}]\}$$

is the least σ-ideal containing W_0, see [11].

(ii) We may directly regard $(\Omega, \mathcal{F}, P)$ as a measure space and perform the usual construction to get the corresponding measure algebra with respect to the σ-ideal of fuzzy sets of P-measure zero, i.e. $I_p := \{a \in \mathcal{F} : P(a) = 0\}$, and $\mathbb{B}_{I_p} := \mathcal{F}/I_p$.

(iii) In [19], [20] another approach is introduced. Let

$$K_0(\mathcal{F}) := \sigma\{H(a) : a \in \mathcal{F}\}$$

be the ordinary σ-algebra generated by the collection $\{H(a) : a \in \mathcal{F}\}$, and define the map

$$H(\cdot) : \mathcal{F} \to K_0(\mathcal{F});$$
$$a \mapsto H(a).$$

The σ-algebra $K_0(\mathcal{F})$ contains also sets of the form $L(a)$ & $M(a)$, $a \in \mathcal{F}$.
Put

$$\mathcal{J}: = \{A \in K_0(\mathcal{F}): (\exists a \in \mathcal{F})[A \subseteq M(a)]\}.$$

Then $\mathcal{J}$ is a σ-ideal in $K_0(\mathcal{F})$ and $\mathbb{B}: = K_0(\mathcal{F})/\mathcal{J}$ is a Boolean σ-algebra, moreover, $K_0(\mathcal{F})/\mathcal{J} = \mathcal{F})/I_0$, see also [14].
In [20] in order to get Boolean representations of an F-quantum space, in which the corresponding σ-homomorphism should respect not only the fuzzy observables but also states on $\mathcal{F}$, minimal and maximal Boolean representation are introduced,

$$\mathbb{B}_{min} := \mathcal{F}/I_Z, \qquad \mathbb{B}_{max} := \mathcal{F}/I_A,$$

where I_A is the intersection of all σ-ideals and

$$I_A = \{a \in \mathcal{F}: H(a) \in \mathcal{F}\}$$

and $I_Z = \bigcap_{s \in S(\mathcal{F})} s^{-1}(0)$, where $s(\mathcal{F})$ is the set of states on $\mathcal{F}$.
Also $\mathbb{B}_{max} = K_0(\mathcal{F})/\mathcal{J}$ and $h_{\mathcal{J}} \circ H$ is a maximal Boolean representation, where $h_{\mathcal{J}}$ is the canonical mapping $h_{\mathcal{J}} : K_0(\mathcal{F}) \to K_0(\mathcal{F})/\mathcal{J}$.

(iv) In [23], [10], an ordinary σ-field of subsets of Ω is introduced

$$K(\mathcal{F}): = \{A \subseteq \Omega: (\exists a \in \mathcal{F})[a > \tfrac{1}{2}] \subseteq A \subseteq [a \geq \tfrac{1}{2}]\}$$

where

$$[a > \tfrac{1}{2}] := \{\omega \in \Omega: a(\omega) > \tfrac{1}{2}\},$$

similarly for $[a \geq \tfrac{1}{2}]$. Let also,

$$I_0: = \{a \in \mathcal{F}: (\exists c \in W_1)[a \wedge c \in W_0]\}$$

and $\mathbb{B}(I_0): = \mathcal{F}/I_0$, is a Boolean σ-algebra and by the soft Loomis-Sikorski Theorem of Dvurečenskij [10], there is an onto σ-homomorphism

$$h_0: K(\mathcal{F}) \to \mathbb{B}(I_0).$$

In [18] there is given a kind of structure of soft fuzzy σ-algebras.
For $u \in W_1$ let

$$\mathcal{F}_u := \{a \in \mathcal{F}: a \vee a' = u\} \cup \{0_\Omega, 1_\Omega\}.$$

then $\mathcal{F}_u$ is a soft fuzzy σ-algebra which induces a corresponding crisp σ-subalgebra $K(\mathcal{F}_u)$.
We have:

$$\mathcal{F} = \bigcup_{u \in W_1} \mathcal{F}_u \quad \text{and} \quad K(\mathcal{F}) = \bigcup_{u \in W_1} K(\mathcal{F}_u).$$

Furthermore we have that for

$$u, v \in W_1\,,\ u \leq v\,, \quad K(\mathcal{F}_u) \supseteq K(\mathcal{F}_v).$$

Now if we set $1_{K(\mathcal{F})} := K(\mathcal{F})$ and $0_{k(\mathcal{F})} := \bigcap_{u \in W_1} K(\mathcal{F}_u)$, then the function

$$c(\cdot) : W_1 \to \mathcal{P}(K(\mathcal{F}));$$
$$u \mapsto c(u) := \bigcup_{u<s} K(\mathcal{F}_s),$$

has the properties of a $\mathcal{P}(K(\mathcal{F}))$-Boolean valued cut in $K(\mathcal{F})$, this can be given an interpretation that $\{c(u) : u \in W\}$ forms a "continuum" of subalgebras.

All the above are byproducts towards the main objective:
To reduce the theory of soft fuzzy probability spaces $(\Omega, \mathcal{F}, P)$ to the theory of ordinary probability theory. Since the soft Boolean algebra $\mathbb{B}_{I_P}$ is complete, one may reduce the theory of soft fuzzy probability spaces $(\Omega, \mathcal{F}, P)$ to a point free probability algebra $(\mathbb{B}_{I_P}, p)$.

Using point-free representations of F-quantum spaces and the following proposition in [4], one can easily see that F-Quantum spaces can be reduced to point-free probability theory.

2.3 Proposition. *An ortholattice $\mathcal{L}$ is a Boolean algebra iff*

$$x \leq y^* \iff x \wedge y = 0$$

for all $x, y \in \mathcal{L}$.

The only difference in the F-quantum case is that $x \wedge y \in W_0$. But using point-free probability spaces this also disappears.

3. Boolean Representation of F-Quantum Spaces

First we will recall some concepts and results from [8].

3.1 DEFINITION. Let $\mathbb{B}$ be a cBa, then the functions

$$l(\cdot)\colon \mathbb{R} \to \mathbb{B}, \qquad u(\cdot)\colon \mathbb{R} \to \mathbb{B}$$

are called $\mathbb{B}$-valued Dedekind cuts iff they satisfy the following properties:

(i) $\bigvee_{x\in\mathbb{R}} c(x) = 1_{\mathbb{B}}$, $\bigwedge_{x\in\mathbb{R}} c(x) = 0_{\mathbb{B}}$, where $c = u$ or l

(ii) $u(x) = \bigvee_{x<y} u(y)$ [order continuity from the right]

and,

$l(x) = \bigvee_{y<x} l(y)$ [order continuity from the left]

since Q is dense in $\mathbb{R}$, then by (ii) we may restrict the supremums to Q.

Let $\mathcal{E}$ be the elementary stochastic space with respect to a complete Boolean algebra $\mathbb{B}$, see [16]. In [6] it is proved that $\mathcal{E}$ is isomorphic to the Boolean powers $\mathbb{R}[\mathbb{B}]$ of the reals and both constitute $\mathbb{B}$-models of the reals, see also [5]. The completion of $\mathcal{E}$ or $\mathbb{R}[\mathbb{B}]$ will be denoted by $\mathcal{V}$, and is called the stochastic space over $\mathbb{B}$. Elements $X \in \mathcal{V}$ will be called random variables. $\mathcal{E}$ and $\mathbb{R}[\mathbb{B}]$ are o-dense in $\mathcal{V}$. Let $(\Omega, \mathcal{A}, P)$ be any probability space, which represents set-theoretically the probability algebra $(\mathbb{B}, p)$. Then we have the following theorem, see [16].

3.2 THEOREM. *Let $(\mathbb{B}, P)$ be a probability σ-algebra and $(\Omega, \mathcal{A}, P)$ be any probability space, which represents set-theoretically $(\mathbb{B}, p)$. Let $\mathcal{V}$ be the stochastic space over $(\mathbb{B}, p)$ and $\mathcal{M}$ be the set of all $\mathcal{A}$-measurable real-valued functions defined on Ω.*

Then every r.v $X \in \mathcal{V}$ is characterized by one of the following:

(i) *A class of almost everywhere equal elements of $\mathcal{M}$,*

(ii) *A $\mathbb{B}$-valued Dedekind cut,*

(iii) *A σ-homomorphism $h\colon \mathbb{B}(\mathbb{R}) \to \mathbb{B}$.*

From now on we suppose that $\mathbb{B}_s$ is a complete soft Boolean σ-algebra and $\gamma\colon \mathcal{F} \to \mathbb{B}_s$ is the canonical σ-homomorphism. Let $x\colon \mathbb{B}(\mathbb{R}) \to \mathcal{F}$ be a fuzzy observable, i.e., an almost σ-homomorphism, then since $\mathbb{B}(\mathbb{R}) = \sigma\{\mathcal{C}\}$, where $\mathcal{C} = \{(x, \infty)\colon x \in \mathbb{R}\}$, we may have the following diagram:

$$\begin{array}{ccc}
\mathcal{F} & \xleftarrow{X^{-1}} & \mathcal{B} \\
\Big\downarrow \gamma & & \Big\uparrow i \\
 & & \mathcal{C} \\
 & & \Big\uparrow \lambda \\
\mathbb{B}_s & \xleftarrow[c]{} & \mathbb{R}
\end{array}$$

where λ is the isomorphism which identifies each real $x \in \mathbb{R}$, with the cut $u(x) := (x, \infty)$.

Then since $x(\cdot)$ and $\gamma(\cdot)$ are σ-homomorphisms we have that $c := \gamma \circ x \circ i \circ \lambda$ have the following properties:

(i) $\bigvee_{r \in \mathbb{R}} c(r) = 1_{\mathbb{B}_s}$, $\quad \bigwedge_{r \in R} c(r) = 0_{\mathbb{B}_s}$,

(ii) $c(r) = \bigvee_{r<s} c(s)$.

that is $c(\cdot)$ is a Boolean-valued Dedekind cut, or a Boolean-valued "real number".

Conversely, if $(\mathbb{B}_s, p)$ is a probability σ-algebra and

$$c(\cdot) : \mathbb{R} \to \mathbb{B}_s$$

is a function with properties (i) and (ii), then $c(\cdot)$ determines a fuzzy observable

$$x : \mathbb{B}(\mathbb{R}) \to \mathcal{F}.$$

This is clear from the Basic Representation Theorem (Theorem 3.2). Thus we have proved the following:

3.3 THEOREM. *The set of all P-almost surely equal fuzzy observables on $\mathcal{F}$ is isomorphic to*

(i) $\mathcal{V}$ *the set of all random variables on* $\mathbb{B}$.

(ii) $\mathbb{R}_c[\mathbb{B}]$, *the set of all* $\mathbb{B}$*-Dedekind cuts on* $\mathbb{R}$.

(iii) $\mathcal{H}$ *the set of all* σ*-homomorphisms* $h : \mathcal{B}(\mathbb{R}) \to \mathbb{B}$.

We may regard the above spaces from an external-absolute point of view, in which case $\mathcal{V}$ for example is a vector lattice, see [16]. However we may view these spaces from an internal-local point of view, and using some $\mathbb{B}$-syntactic methods we manage to see $\mathcal{V}$ or $\mathbb{R}_c[\mathbb{B}]$ as a set of Boolean n-valued reals. In particular the Boolean power $\mathbb{R}[\mathbb{B}]$ is dense in $\mathbb{R}_c[\mathbb{B}]$, see [6].

On $\mathbb{R}_c[\mathbb{B}]$ we may define a structure of "real numbers". If $f, g \in \mathbb{R}_c[\mathbb{B}]$ we define

$$\|f \leq g\| := \bigvee_{\substack{x,y\in\mathbb{R}:\\ x\leq y}} [f(x) \wedge g(y)]$$

$$(f+g)(z) := \bigvee_{\substack{x,y\in\mathbb{R}:\\ x+y=z}} [f(x) \wedge g(y)]$$

$$= \bigvee_{x\in\mathbb{R}} [f(x) \wedge g(z-x)]. \qquad (*)$$

It is worth noting in [10, Theorem 3.1] that Boolean-valued Dedekind cuts are used and the sum of two observables is defined through the sum of the corresponding Boolean-Dedekind cuts, exactly like in $(*)$, see also [14]. We may use the denseness of $\mathbb{R}[\mathbb{B}]$ in $\mathbb{R}_c[\mathbb{B}]$, and transfer the whole real analysis into a Boolean analysis of observables. From an external absolute point of view, we may follow for example [21] to develop a whole calculus of fuzzy observables.

For discrete fuzzy observables Theorem 3.3 takes the following form:

3.4 Theorem. *The set of all P-almost surely equal elementary fuzzy observables on $\mathcal{F}$ is isomorphic to:*

(i) *The elementary stochastic space $\mathcal{E}$ on $\mathbb{B}_s$*
(ii) *The Boolean power $\mathbb{R}[\mathbb{B}_s]$, of $\mathbb{R}$.*

To each $X \in \mathcal{V}$ and so to each fuzzy observable we may also give an integral representation, see [16]. The *spectrum* of X in $\mathcal{V}$ is defined as follows:

$$I^X(\cdot)\colon \mathbb{R} \to \mathcal{J}(\mathbb{B}) \subseteq \mathcal{V};$$
$$\xi \mapsto I^X(\xi) := I_{s_X(\xi)}(\cdot),$$

where

$$I_{s_X(\xi)}(t) := \begin{cases} 1 & \text{if } t = s_X(\xi), \\ 0 & \text{otherwise}, \end{cases}$$

and

$$s_X(\xi) := [X < \xi], \qquad \xi \in \mathbb{R}.$$

Then we have,

$$X = \int_{-\infty}^{+\infty} \xi \, \mathrm{d}I^X(\xi).$$

For details see [16].

Now that the theory of F-Quantum spaces seems to be reducible to the theory of Boolean algebras, one may argue that soft fuzzy σ-algebras loose importance for quantum theories, see also [19]. However the true lesson from the above reduction should be that the Boolean models for the reals should replace the classical model of $\mathbb{R}$, as a model for quantum measurements. Thus starting from nonstandard models of the real numbers, one can build up a non-Cantorian model for quantum mechanics, parallel in many respects to the absolute-Cantorian model of Birkhoff-von Neumann. In the next section, we shall indicate the general lines towards to such a development.

4. The Theory of Orthospaces

In a series of papers [1], [2], [3], Bell has developed a theory of orthospaces based on a proximity or tolerance relations. Let us first review some of the concepts and results.

4.1 Definition. A *proximity structure (or tolerance structure)*, $(X, \approx)$ is a pair, where X is a set and $\approx$ is a binary relation which is symmetric and reflexive, i.e.

(i) $x \approx x$ for all $x \in X$,
(ii) $x \approx y = y \approx x, \quad x, y \in X$.

In general "$\approx$" is not transitive. If it is, then $\approx$ is called a *similarity relation.* These relations play an important role in the theory of fuzzy relations, see [22].

The dual of $(X, \approx)$ denoted as $(X, \perp)$ is known as an *orthogonality space* see [15]. The relation "$\perp$" is the set-theoretic complement of $\approx$ in $X \times X$, and conversely, thus,

$$x \perp y \qquad \Longleftrightarrow \qquad x \not\approx y.$$

Usually in an orthogonality space, we also include a bottom element 0, in symbols we have $(X, \perp, 0)$.

An *orthospace* $(X, \leq, \perp, 0)$ is a structure such that $(X, \leq)$ is a preordered set ($\leq$ is only reflexive and transitive), with least elements 0, and $(X, \perp, 0)$ is an orthogonality space such that for all $x, y \in X$

$$x \leq y \ \& \ y \perp z \ \rightarrow x \perp z.$$

4.2 Definition. Let $(X, \approx)$ be a proximity space. For every $x \in X$, we define the *quantum* at x to be

$$Q_x := \{y \in X : x \approx y\}.$$

Zeeman [24] suggested that the quantum Q_x should be considered as made up from all indistinguishable to x elements. Thus we perceive Q_x as a "plot" or "monad", X may be viewed as the set of outcomes of experiment, or the set of states of a quantum system and $\approx$ as a relation of equality up to the limits of experimental error. Thus Q_x is the "outcome within a specified margin of error" of experimental practice.

4.3 EXAMPLE. (i) If $^*\mathbb{R}$ is the Robinsonian nonstandard set of reals, then the monads of $^*\mathbb{R}$ with respect to the similarity relation of "infinitely close", can be taken as quanta, and $x \perp y$ iff $x \not\approx y$, that is x is orthogonal to y iff x and y belong to different monads.

(ii) If H is a Hilbert space, and $X := H - \{0\}$, then

$$s \approx t :\leftrightarrow \; <s,t> \neq 0 \qquad \text{(i.e., } s \text{ is not orthogonal to } t\text{)},$$

then "$\approx$" is a proximity relation.

(iii) If $\mathcal{F}$ is an F-quantum space then, for $a, b \in \mathcal{F}$,

$$a \approx b \qquad \Longleftrightarrow \qquad a \not\perp b \qquad \Longleftrightarrow \qquad a + b > 1$$

and

$$a \approx_F b \qquad \Longleftrightarrow \qquad a \not\perp_F b \qquad a \wedge b > \frac{1}{2}.$$

4.4 DEFINITION. Let $A \subseteq X$ be a classical subset of X. However, due to the presence of proximity relation, we may define a non-classical part of X as follows:

$$\bigcup_{x \in A} Q_x ,$$

and

$$\text{Parts}(X): \; = \{ \bigcup_{x \in A} Q_x : A \in \mathcal{P}(X) \}.$$

This definition can be generalized by taking instead of $\mathcal{P}$ a σ-field of subset of X.

4.5 THEOREM. [1] *The family Parts(X) of parts of a proximity structure $(X, \approx)$, forms a complete ortholattice, under set inclusion, set union and intersection, as supremum and infimum correspondingly and in which the orthocomplement $A^\perp$ of $A \in$ parts(X) is defined as the parts of X, which are "outside" of X, i.e.*

$$A^\perp := \bigcup_{y \notin A} Q_y = \{ x \in X : (\exists y \in X)[y \notin A \;\&\; x \approx y] \}.$$

4.6 THEOREM. [1] *Any ortholattice L is completely embeddable in a complete ortholattice of Parts(X) of a proximity space $(X, \approx)$.*

4.7 THEOREM. [1] [15] *Each complete ortholattice L is isomorphic to one of the form Parts(X), for some proximity space $(X, \approx)$.*

4.8 EXAMPLE. In the Hilbert space example, the $*$-lattice of parts of $X = H \setminus \{0\}$ is isomorphic to the $*$-ortholattice of closed subspaces of H. Consequently $*$-lattices of parts of proximity spaces include the $*$-lattices of closed subspaces of Hilbert spaces, i.e., the Birkhoff-von Neumann's Quantum logic.

According to Bell, the topologically valid formulas coincide with the tautologies of intuitionistic logic, the formulas associated with discrete spaces with the tautologies of classical logic, and the proximity valid formulas as the tautologies of quantum mechanics.

5. Soft Boolean Powers

In this section using the previous concepts, we shall try to give a non-Cantorian analogue of the Hilbert space models.

Let $\mathbb{B}$ be a complete soft Boolean algebra, and consider the Boolean power $\mathbb{R}[\mathbb{B}]$. On $\mathbb{R}[\mathbb{B}]$ we have:

(i) $\|f = f\| = 1_{\mathbb{B}}$,

(ii) $\|f = g\| \;=\; \|g = f\|$,

(iii) $\|f = g\| \cdot \|g = h\| \leq \|f = g\|$.

So that the relation $f =_{\mathbb{B}} g$ iff $\|f = g\| = 1_{\mathbb{B}}$ is a similarity on $\mathbb{R}[\mathbb{B}]$. However since $\|f = g\| = 1_{\mathbb{B}}$ iff $f = g$ as function, the equivalent classes,

$$[f] := \{g \in \mathbb{R}[\mathbb{B}] : f =_{\mathbb{B}} g\}$$

contain exactly one element.

In order to construct some structure of the form, Parts$(\mathbb{R}[\mathbb{B}])$, we need some quantum that contains more than one element. To this end consider the Boolean ultrapower $\mathbb{R}[\mathbb{B}]/\approx_U$, with respect to a δ-incomplete ultrafilter U see [17]. Then ${}^*\mathbb{R} := \mathbb{R}[\mathbb{B}]/\approx_U$ is a model for the reals which contain stochastic infinitesimals (equivalent classes of elementary random variables), and which generalizes the usual models of infinitesimal analysis. The "infinitely close" relation "$\approx$" is a similarity relation and thus we may construct the structure

$$\left(\text{Parts}({}^*(\mathbb{R})),\ \subseteq,\ \cup,\ \cap,\ {}^\perp\right),$$

which according to Theorem 4.5 is a complete ortholattice. This structure can be regarded as a non-Cantorian framework for quantum logics. In forming Parts($^*\mathbb{R}$), we may use instead of $\mathcal{P}(^*\mathbb{R})$ either the σ-algebra generated by the internal subsets, or the algebra of the internal subsets themselves.

Finally the last possibility is to use instead of an δ-incomplete ultrafilter, which giveS rise to a similarity relation on $^*\mathbb{R}$, a "tolerance ultrafilter" U, as one might call it, that is $U \subseteq \mathbb{B}$ such that:

(i) $1_{\mathbb{B}} \in U$ & $0_{\mathbb{B}} \in U$,
(ii) $a, b \in U \implies a \wedge b \in U$.

Then the relation induced by this U is a tolerance relation, and we may proceed to construct again the ortholattice of parts. This will give a more pure quantum logic model.

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