Tatra Mountains Math. Publ., 1 (1992), 135–140

TATRA MOUNTAINS
Mathematical Publications

ON A TYPE OF ENTROPY OF DYNAMICAL SYSTEMS

BELOSLAV RIEČAN

This contribution has three aims. First we compare the concept by D. Dumitrescu [2], [3] (see also [11]) with that by P. Maličký and the author [5]. Secondly we present two counting formulas for the entropy [5]. Finally we present some remarks concerning the fuzzy entropy and especially we repeat the suggestion of P. Maličký to define a very close but different invariant for fuzzy dynamical systems.

1. Measure preserving transformation and the entropy of a fuzzy partition

Following [2] and [3] we shall use the following definitions.

A family $\mathcal{F} \subset \langle 0,1\rangle^X$ of fuzzy subsets of a set X is said to be a σ-algebra, if the following axioms are satisfied:

(i) $1_X \in \mathcal{F}$.
(ii) If $f, g \in \mathcal{F}$, then $f \cdot g \in \mathcal{F}$, $f - g \in \mathcal{F}$, where $f \cdot g(x) = f(x)g(x)$, $f - g(x) = \max(f(x) - g(x), 0)$.
(iii) If $f_n \in \mathcal{F}$ $(n = 1,2,\dots)$, then $\bigcup_{n=1}^{\infty} f_n \in \mathcal{F}$, where $\bigcup_{n=1}^{\infty} f_n(x) = \min\left(\sum_{n=1}^{\infty} f_n(x), 1\right)$.

A function $m : \mathcal{F} \to \langle 0, \infty)$ is called a fuzzy measure, if

(i) $m(0_X) = 0$.
(ii) $m\left(\bigcup_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} m(f_n)$, whenever $f_n \in \mathcal{F}$ $(n = 1,2,\dots)$ and $\sum_{n=1}^{\infty} f_n \le 1$.

A fuzzy partition is a finite collection $\mathcal{A} = \{f_1, \dots, f_n\}$ of members of $\mathcal{F}$ such that $\sum_{i=1}^{n} f_i(x) = 1$ for all $x \in X$.

A fuzzy partition $\mathcal{A} = \{f_1, \dots, f_n\}$ is a refinement of a fuzzy partition $\mathcal{B} = \{g_1, \dots, g_m\}$, if there are disjoint sets $I(1), \dots, I(m) \subset \{1, \dots, n\}$ such that

$$g_i = \sum_{j \in I(i)} f_j,$$

for every $i = 1, \dots, m$. If $\mathcal{A}$ is a refinement of $\mathcal{B}$, we write $\mathcal{A} \geqq \mathcal{B}$.

A common refinement of two fuzzy partitions $\mathcal{A} = \{f_1, \dots, f_n\}$, $\mathcal{B} = \{g_1, \dots, g_m\}$ is the collection

$$\mathcal{A} \vee \mathcal{B} = \{f_i \cdot g_j : i = 1, \dots, n, \ j = 1, \dots, m\}.$$

(Of course, $\mathcal{A} \vee \mathcal{B} \geq \mathcal{A}$, $\mathcal{A} \vee \mathcal{B} \geq \mathcal{B}$.)

A transformation $T : X \to X$ is called measure preserving, if the following implication holds:

$$f \in \mathcal{F} \implies f \circ T \in \mathcal{F}, \quad m(f \circ T) = m(f).$$

If $\mathcal{A} = \{f_1, \dots, f_n\}$ is a fuzzy partition, then we define

$$T^{-1} = \{f_1 \circ T, \dots, f_n \circ T\}.$$

If $\mathcal{A} = \{f_1, \dots, f_n\}$ is a fuzzy partition, then its entropy $H(\mathcal{A})$ is defined by the formula

$$H(\mathcal{A}) = \sum_{i=1}^{n} \varphi(m(f_i)),$$

where $\varphi : \langle 0, 1 \rangle \to R$ is defined by $\varphi(x) = -x \log x$ $(x > 0)$, $\varphi(0) = 0$.

Now if $\mathcal{A}$ is a fuzzy partition, then we define

$$H_n(\mathcal{A}) = H(\mathcal{A} \vee T^{-1}\mathcal{A} \vee \dots \vee T^{-(n-1)}\mathcal{A}),$$

$$h(\mathcal{A}, T) = \lim_{n \to \infty} \frac{1}{n} H_n(\mathcal{A}),$$

$$h(T) = \sup\{h(\mathcal{A}, T);\ \mathcal{A} \text{ is a fuzzy partition}\}.$$

The previous definition is taken from [3]. Of course, the same definition has been presented in [5] in a more special case, where $\mathcal{F}$ is a set of all measurable functions (with respect to a σ-algebra $\mathcal{S} \subset 2^X$) and $m(f) = \int f \, \mathrm{d}P$, where $P \colon \mathcal{S} \to \langle 0, 1 \rangle$ is a probability measure.

2. An alternative characterization of entropy

First we shall repeat some properties of the entropy $H(\mathcal{A})$ and the conditional entropy

$$H(\mathcal{A} \mid \mathcal{B}) = \sum_i \sum_j m(g_j)\varphi\left(\frac{m(f_i \cdot g_j)}{m(g_j)}\right),$$

where the sum is taken over all j for which $m(g_j) > 0$.

PROPOSITION 1. *If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are an arbitrary fuzzy partition, then*

(i) $H(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{B}) + H(\mathcal{A} \mid \mathcal{B})$,

(ii) $\mathcal{A} \leq \mathcal{B} \implies H(\mathcal{A} \mid \mathcal{C}) \leq H(\mathcal{B} \mid \mathcal{C})$,

(iii) $\mathcal{A} \leq \mathcal{B} \implies H(\mathcal{C} \mid \mathcal{A}) \geqq H(\mathcal{C} \mid \mathcal{B})$.

THEOREM 1. $h(\mathcal{A}, T) = \lim_{n\to\infty} H\left(\mathcal{A} \mid \bigvee_{i=1}^{n} T^{-1}\mathcal{A}\right)$.

Proof. By Proposition 1 (i) we obtain

$$H\left(\bigvee_{i=0}^{k} T^{-i}\mathcal{A}\right) = H\left(\mathcal{A} \vee T^{-1}\left(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{A}\right)\right) =$$

$$= H\left(T^{-1}\left(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{A}\right)\right) + H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-i}\mathcal{A}\right) =$$

$$= H\left(\bigvee_{i=0}^{k-1} T^{-i}\mathcal{A}\right) + H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-i}\mathcal{A}\right).$$

Now by induction we obtain

$$\text{(iv)} \quad H\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}\right) = H(\mathcal{A}) + \sum_{k=1}^{n-1} H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-i}\mathcal{A}\right).$$

By (iii) we obtain that $H\left(\mathcal{A} \mid \bigvee_{i=1}^{n} T^{-1}\mathcal{A}\right)$ is decreasing, so that

$$\lim_{n\to\infty} H\left(\mathcal{A} \mid \bigvee_{i=1}^{n} T^{-1}\mathcal{A}\right)$$

exists. But then there exists also the limit of the Cesaro means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-1}\mathcal{A}\right).$$

By (iv) we obtain

$$\lim_{n\to\infty} H\left(\mathcal{A} \mid \bigvee_{i=1}^{n} T^{-1}\mathcal{A}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-i}\mathcal{A}\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(H(\mathcal{A}) + \sum_{l=1}^{n-1} H\left(\mathcal{A} \mid \bigvee_{i=1}^{k} T^{-i}\mathcal{A}\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}\right) = h(\mathcal{A}, T).$$

□

Entropy and generators

The classical Kolmogorov-Sinaj theorem states that

$$h(T) = h(T, \mathcal{A}),$$

whenever $\mathcal{A}$ is a generator of the dynamical system. A functional (= fuzzy) version of the generator theorem was presented in [5]. Here we want to present a generalization of the theorem in the general case (another modification is contained in [10]). The key is a representation theorem [1]:

PROPOSITION 2. *Denote by $\mathcal{S}$ the family $\{A \subset X\,;\ \mathcal{X}_A \in \mathcal{F}\}$ and define $\mu\colon \mathcal{S} \to \mathcal{R}$ by $\mu(A) = m(\mathcal{X}_A)$. Then every function $f \in \mathcal{F}$ is $\mathcal{S}$-measurable and*

$$m(f) = \int_X f\, d\mu,$$

for every $f \in \mathcal{F}$.

P r o o f. [1], Theorem 4.1. □

According to Proposition 2 the Dumitrescu entropy $h(T)$ (with respect to $X, \mathcal{F}, m, T$) coincides with the Maličký and Riečan entropy $h_G(T)$ (with respect to $X, \mathcal{S}, \mu, T$). Moreover, in [5] an arbitrary set G of functions is considered with $G \subset \mathcal{F}$ and then an invariant $h_G(T)$ is defined by

$$h_G(T) = \sup\{h(\mathcal{A}, T)\,;\ \mathcal{A} \text{ is a finite partition, } \mathcal{A} \subset G\}.$$

Evidently $h_{\mathcal{F}}(T) = h(T)$.

Now as a corollary of a theorem of [5] we can obtain the following theorem for the fuzzy entropy.

THEOREM 2. *If $\mathcal{A} = \{\mathcal{X}_{E_1}, \dots, \mathcal{X}_{E_n}\}$ is a fuzzy partition consisting of crisp sets generating the σ-algebra $\mathcal{S}$, then*

$$h_G(T) \le h(\mathcal{A}, T) + L_G ,$$

where

$$L_G = \sup\Big\{m\Big(\sum_{j=1}^{n} \varphi(g_j)\Big);\ \{g_1, \dots, g_n\} \subset G \text{ is a fuzzy partition}\Big\}.$$

P r o o f . By Proposition 2 there exists $\mu : \mathcal{S} \to \mathcal{R}$ such that

$$m(g) = \int_X g \, \mathrm{d}\mu .$$

Now it is possible to use Theorem 1 from [5]:

$$h_G(T) \le h(\mathcal{A}, T) + K_G ,$$

where

$$K_G = \sup\Big\{\int \sum_{j=1}^{n} \varphi(g_j) \mathrm{d}\mu ;\ \{g_1, \dots, g_n\} \text{ is a fuzzy partition}, \quad g_j \in G\Big\}.$$

Evidently $K_G = L_G$. □

Concluding remarks

The generalization of the Kolmogorov–Sinaj invariant due to Dumitrescu, Maličký and the author has the following two advantages:

1. It can have a positive value also in some cases, when the Kolmogorov entropy is zero, hence the new invariant could better distinguish non-isomorphic dynamical systems.

2. It is applicable also for the case of fuzzy dynamical systems.

Of course, in many cases $h_G(T) = \infty$, e.g. if G contains all constant fuzzy sets.

3. As a solution of this problem P. Maličký in [5] suggested the following modification: instead of $H(\mathcal{A} \vee T^{-1}\mathcal{A} \vee T^{-(n-i)}\mathcal{A})$ to consider

$$H_n(\mathcal{A}) = \inf\{H \mid \mathcal{C});\ \mathcal{C} \ge \mathcal{A},\ \mathcal{C} \ge T^{-1}\mathcal{A}, \dots, \mathcal{C} \ge T^{-n-1}\mathcal{A}\}.$$

4. Another solution of the preceding problem has been suggested by T. Hudetz in [4].

5. Another modification, we suggested in [5], is to consider a Markov operator $U : \mathcal{F} \to \mathcal{F}$ instead of the special case $U = U_T$ defined by $U_T(f) = f \circ T$.

REFERENCES

[1] BUTNARIU, D.—KLEMENT, E. P.: *Triangular norm-based measures and their Markov kernel representation*, J. Math. Anal. Appl. **162** (1991), 111–143.

[2] DUMITRESCU, D.: *Measure preserving transformation and the entropy of fuzzy partition*, In: 13th Linz seminar on Fuzzy set Theory (Linz 1991), pp. 25–27.

[3] DUMITRESCU, D.: *Fuzzy measures and the entropy of fuzzy partitions.*, J. Math. Anal. Appl. (to appear).

[4] HUDETZ, T.: *Entropy and dynamical entropy for continuous fuzzy partitions*, J. Math. Anal. Appl. (to appear).

[5] MALIČKÝ, P.—RIEČAN, B.: *On the entropy of dynamical systems*, In: Proc. Ergodic Theory and Related Topics II. (Georgenthal 1986), Teubner, Berlin, 1987, pp. 135–138.

[6] MARKECHOVÁ, D.: *The entropy on F-quantum spaces*, Math. Slovaca **40** (1990), 177–190.

[7] MARKECHOVÁ, D.: *The entropy of fuzzy dynamical systems and generators*, Fuzzy Sets and Systems (to appear).

[8] MESIAR, R.: *The Bayes formula and the entropy on fuzzy probability spaces*, Int. J. General Systems **20** (1991), 67–71.

[9] NOVÁK, V.: Fuzzy Sets and their Applications, Hilger, Bristol, 1989.

[10] RIEČAN, B.: *On the entropy and generators of dynamical systems*, Appl. Math. (to appear).

[11] RYBÁRIK, J.: *The entropy of Q – F-dynamical system*, BUSEFAL **48** (1991), 24–26.

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
CZECHO-SLOVAKIA