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ON FOURIER—STIELTJES TRANSFORMS IN VECTOR LATTICES

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. This paper is concerned with a characterization of finite Fourier–Stieltjes transforms of some functions taking their values in vector lattices. As for the terminology and some results of ordered spaces we make use of [1].

1. Preliminaries

Let Y be a (Dedekind) complete vector lattice. Denote by $L^o(X,Y)$ the vector space of all o-bounded operators on the normed space X into Y, that is, if $U \in L^o(X,Y)$, then $\{U(x); \|x\| \le 1\}$ is an o-bounded subset of Y. For $U \in L^o(X,Y)$ we put

$$\|U\| = \sup\{|U(x)|\,;\, \|x\| \le 1\}\,.$$

Let T be a finite closed interval of the real line and let $C(T)$ denote the space of all scalar continuous functions on T with the usual sup norm. If $U \in L^o(C(T),Y)$, $T = [0, 2\pi]$, then an element of Y of the form

$$\hat{U}(n) = U(e^{-int}),$$

is called the n-th Fourier coefficient of U. The (formal) series

$$\sum_{n\in\mathbb{Z}} \hat{U}(n) e^{inx}$$

is called the Fourier series of U. It is clear that there exists an element $0 \le C \in Y$ such that

$$\left|\hat{U}(n)\right| \le C, \qquad n \in \mathbb{Z}.$$

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In the following let $\mathbb{T}$ denote the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ and $\mathbb{Z}$ denoting the additive group of reals, integers, respectively), as a model we may think of the interval $[0, 2\pi)$. A trigonometric polynomial on $\mathbb{T}$ is a function $a = a(t)$ defined on $\mathbb{T}$ by $a(t) = \sum_{-n}^{n} a_j e^{ijt}$. Denote by $p(\mathbb{T})$ the set of all trigonometric polynomials on $\mathbb{T}$. We shall need the following theorem [3, Th. 2.12] asserting that trigonometric polynomials are dense in $C(\mathbb{T})$.

THEOREM A. *For every $f \in C(\mathbb{T})$ we have $\sigma_n(f) \to f$, $n \to \infty$, in the $C(\mathbb{T})$ norm.*

Recall that

$$\sigma_n(f,t) = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)\, e^{ijt},$$

where $\hat{f}(j)$ is the j-th Fourier-Lebesgue coefficient of f defined by

$$\hat{f}(j) = \frac{1}{2\pi} \int f(t)\, e^{-ijt}\, \mathrm{d}t.$$

(The integration is taken over $\mathbb{T}$.)

The following simple lemma will be useful for us.

LEMMA. *Let $U : C(\mathbb{T}) \to Y$ be an o-bounded linear mapping. For every $a = \sum_{-n}^{n} a_j e^{ijt}$ we have $U(a) = \sum_{-n}^{n} a_j \hat{U}(-j)$ and $|U(a)| \le \|a\|\,\|U\|$, where*

$$\|a\| = \sup_t |a(t)|.$$

We have the following result.

THEOREM 1 (Parseval's formula). *Let $f \in C(\mathbb{T})$ and $U \in L^o(C(\mathbb{T}), Y)$. Then*

$$U(f) = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)\, \hat{U}(-j).$$

Proof. Since $f = \lim_{n\to\infty} \sigma_n(f)$ in the $C(\mathbb{T})$ norm, it follows from lemma and the fact that U is o-bounded (hence o-continuous) that

$$U(f) = U\big(\lim_{n\to\infty} \sigma_n(f)\big) = \lim_{n\to\infty} U\big(\sigma_n(f)\big) =$$
$$= \lim_{n\to\infty} \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)\, \hat{U}(-j).$$

□

Remark. The fact that the preceding limit exists is an implicit part of the theorem. It is equivalent to the C-1 (Cesàro) summability of the series $\sum \hat{f}(j)\hat{U}(-j)$, the members of which are elements of the space Y. If this last series converges then clearly

$$U(f) = \sum_{-\infty}^{\infty} \hat{f}(j)\,\hat{U}(-j).$$

COROLLARY (Uniqueness theorem). *If $\hat{U}(j) = 0$ for all $j \in \mathbb{Z}$, then $U = 0$.*

Parseval's formula enables us to characterize sequences of Fourier coefficients of o-bounded linear operators on $C(\mathbf{T})$ similar to the case of linear functionals ([3, 7.3] or [2]).

THEOREM 2. *Let (y_j) be a two-way sequence of elements of Y. Then the following two conditions are equivalent:*

(a) *There is a mapping $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \le C \in Y$ such that $\hat{U}(j) = y_j$ for all $j \in \mathbb{Z}$.*

(b) *For all trigonometric polynomials $a = \sum_{-l}^{l} a_j e^{ijt}$ there holds*

$$\left|\sum_{-l}^{l} a_{-j} y_j\right| \le \|a\|\, C \ \text{ with } 0 \le C \in Y.$$

Proof. Clearly (a) implies (b) since

$$\left|\sum_{-l}^{l} a_{-j} y_j\right| = \left|\sum_{-l}^{l} a_{-j}\hat{U}(j)\right| =$$

$$= \left|\sum_{-l}^{l} a_{-j} U(e^{-ijt})\right| \le \|U\| \,.\sup_t \left|\sum_{-l}^{l} a_{-j} e^{-ijt}\right| \le C\,\|a\|\,.$$

Conversely let for $\{y_j\} \subset Y$ for some $C \in Y$

$$\left|\sum_{-l}^{l} a_{-j} y_j\right| \le C \sup_t \left|\sum_{-l}^{l} a_{-j} e^{-ijt}\right|.$$

Put

$$U\left(\sum_{-l}^{l} a_j e^{ijt}\right) = \sum_{-l}^{l} a_{-j} y_j\,.$$

Then

$$\left|U(\sum_{-l}^{l} a_{-j}e^{-ijt})\right| \le C \sup_t \left|\sum_{-l}^{l} a_{-j}e^{-ijt}\right| .$$

It follows that U is an o-bounded operator on trigonometric polynomials, these are dense in $C(\mathbf{T})$, hence U has an o-bounded extension to $C(\mathbf{T})$. Also we obtain $\hat{U}(j) = y_j$. □

Let (y_j) be a two-way sequence of elements of Y. Put

$$\sigma_N(Y,t) = \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) y_{-j}e^{-ijt}, \qquad N = 1,2,\dots$$

and denote by $S_N(Y)$ the o-bounded linear mapping on $C(\mathbf{T})$ defined by

$$S_N(Y)(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t)\sigma_N(Y,t)\,\mathrm{d}t, \qquad f \in C(\mathbf{T}), \quad N = 1,2,\dots$$

If $U \in L^o(C(\mathbf{T}),Y)$ and if $y_j = \hat{U}(j)$, we shall write

$$\sigma_N(Y,t) = \sigma_N(U,t) \qquad \text{and} \qquad S_N(Y) = S_N(U).$$

We have

$$S_N(Y)(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t)\sigma_N(Y,t)\,\mathrm{d}t =$$

$$= \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)y_{-j}, \qquad f \in C(\mathbf{T}), \quad N = 1,2,\dots$$

We may now prove the following.

THEOREM 3. *The members of a two-way sequence (y_j) in Y are the Fourier coefficients of some $U \in L^o(C(\mathbf{T}),Y)$, with $\|U\| \le C \in Y$, if and only if $\|S_N(Y)\| \le C$, $N = 1,2,\dots$.*

Proof. The necessity. Let $y_j = \hat{U}(j)$ for some $U \in L^o(C(\mathbf{T}),Y)$ with $\|U\| \le C$. Then $S_N(Y) = S_N(U)$, $N = 1,2,\dots$ Recall that $\|\sigma_N(f)\| \le \|f\|$ for all $f \in C(\mathbf{T})$. Since, for $f \in C(\mathbf{T})$, $S_N(U)(f) = U(\sigma_N(f))$, we have

$$\begin{gathered}\|S_N(Y)\| = \|S_N(U)\| = \sup\{|S_N(U)(f)| : f \in C(\mathbf{T}), \|f\| \le 1\} = \\ \sup\{|(U(\sigma_N(f))| : f \in C(\mathbf{T}), \|f\| \le 1\} \le \\ \sup\{|U(f)| : f \in C(\mathbf{T}), \|f\| \le 1\} = \|U\| \le C,\end{gathered}$$

for $N = 1, 2, \dots$.

The sufficiency. Take $a = \sum_{-l}^{l} a_j e^{ijt}$. Then we have

$$\sum_{-l}^{l} y_{-j} a_j = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) y_{-j} a_j = \lim_{N\to\infty} S_N(Y)(a) .$$

Thus

$$\left| \sum_{-l}^{l} y_{-j} a_j \right| = \lim_{N\to\infty} |S_N(Y)(a)| \le \|a\| \sup_N \|S_N(Y)\| \le \|a\|\, C .$$

According to preceding theorem there exists $U \in L^o(C(\mathbf{T}), Y)$ such that $y_j = \hat{U}(j)$ and $\|U\| \le C$. □

2. Fourier–Stieltjes coefficients of vector functions of o-bounded variation

Recall that a function $\boldsymbol{g}$, defined on T and taking values in Y, is said to be of o-bounded variation, if the set of all elements of the form

$$\sum_j |\boldsymbol{g}(t_{j+1}) - \boldsymbol{g}(t_j)| ,$$

corresponding to all finite partitions of the interval T, is o-bounded. We shall denote by $o\text{-}\operatorname*{var}_{t\in T} \boldsymbol{g}(t)$ the least upper bound of this set.

We shall need the following result [1, 7.1.5].

The general form of the o-bounded linear operator $U : C(T) \to Y$ is given by the formula

$$U(f) = \int_T f(t)\, \mathrm{d}\boldsymbol{g}(t) ,$$

where $\boldsymbol{g} : T \to Y$ is a function of o-bounded variation.

Denote by $BV^o(T, Y)$ the vector space of all functions on T with values in Y of the o-bounded variation. Further if $\boldsymbol{g} \in BV^o(T, Y)$, $T = [0, 2\pi]$, then an element of Y of the form

$$\hat{\boldsymbol{g}}(n) = \frac{1}{2\pi} \int_T e^{-int}\, \mathrm{d}\boldsymbol{g}(t)$$

is called the n-th Fourier–Stieltjes coefficient of $\boldsymbol{g}$.

Now we may reformulate the Theorem 2 in the following form.

THEOREM 4. *Let Y be a complete vector lattice. Let (y_k) be a two-way sequence of elements of Y. Then the following two conditions are equivalent:*

(a) *There is a function $g : T \to Y$ of o-bounded variation with $o\text{-}\operatorname*{var}_{t\in T} g(t) \leq C \in Y$ such that y_j are Fourier–Stieltjes coefficients of $g(t)$, i.e.,*

$$y_j = \hat{g}(j) = \frac{1}{2\pi} \int\limits_T e^{-ijt}\, \mathrm{d}g(t) \quad \text{for all} \qquad j \in \mathbb{Z}.$$

(b) *For all trigonometric polynomials $a = \sum_{-l}^{l} a_j e^{ijt} \in p(T)$ there holds*

$$\left| \sum_{-l}^{l} a_{-j} y_j \right| \leq \|a\|\, C$$

for some $C \in Y$.

Proof. Clearly (a) implies (b) since

$$\left| \sum_{-l}^{l} a_{-j} y_j \right| = \left| \sum_{-l}^{l} a_{-j} \frac{1}{2\pi} \int\limits_T e^{-ijt}\, \mathrm{d}g(t) \right| =$$

$$= \left| \frac{1}{2\pi} \int\limits_T \left(\sum_{-l}^{l} a_{-j} e^{-ijt}\, \mathrm{d}g(t) \right) \right| \leq \|a\| \left[o\text{-}\operatorname*{var}_{t\in T} g(t) \right],$$

by using [1, 7.1.4, Corollary].

If we assume (b), then, since $p(T) = p(\mathbf{T})$, the linear mapping $U : p(T) = p(\mathbf{T}) \to Y$ from the proof of Theorem 2 is an o-bounded linear mapping that admits an extension that is an o-bounded linear mapping on $C(\mathbf{T})$ with $\|U\| \leq C$. By the Stone-Weierstrass theorem $C(\mathbf{T})$ is (uniform) dense in $C(T)$. Hence according to [1, 6.3.3, Proposition 1] U can be extended to $C(T)$ with the same vector norm $\|U\| \leq C \in Y$. But according to [1, 7.1.3, Corollary] there exists a function g of o-bounded variation such that

$$U(f) = \int\limits_T f(t)\, \mathrm{d}g(t), \qquad f \in C(T).$$

Clearly $\hat{U}(j) = \hat{g}(j) = y_j$. □

If $g \in BV^o(T,Y)$, then the (formal) series

$$\sum_{n\in\mathbb{Z}} \hat{g}(n) e^{inx}$$

is called the Fourier–Stieltjes series of g. We may now prove the following.

THEOREM 5. *Let Y be a complete vector lattice. The trigonometric series*

$$\sum_{n\in\mathbb{Z}} y_j e^{inx}, \qquad y_j \in Y,$$

is the Fourier–Stieltjes series of the function g of the o-bounded variation, i.e., $y_j = \hat{g}(j)$, $j \in \mathbb{Z}$, if and only if there exists an element $0 \le C \in Y$ such that

$$\|S_N(Y)\| \le C, \qquad N = 1, 2, \ldots$$

Proof. If there exists a function g of o-bounded variation, $g \in BV^o(T, Y)$ such that $y_j = \hat{g}(j)$, $j \in \mathbb{Z}$, then as we know the equation

$$U(f) = \int_T f(t)\,\mathrm{d}g(t), \qquad f \in C(T),$$

defines an o-bounded linear mapping $U : C(T) \to Y$ with $\|U\| \le C$ for some $0 \le C \in Y$. In particular U is an o-bounded linear mapping on $C(\mathbf{T})$ into Y with same vector norm $\|U\| \le C$. Hence according to Theorem 3 we have

$$\|S_N(Y)\| = \|S_N(U)\| = \|S_N(g)\| \le C, \qquad N = 1, 2, \ldots.$$

Conversely, if $\|S_N(Y)\| \le C$, $N = 1, 2, \ldots$, for some $0 \le C \in Y$, then according to Theorem 3 there exists an o-bounded linear mapping $U : C(\mathbf{T}) \to Y$ such that $\hat{U}(j) = y_j$. By the Stone-Weierstrass theorem U admits an extension to $C(T)$. But then there exists a function g of o-bounded variation such that

$$U(f) = \int_T f(t)\,\mathrm{d}g(t), \qquad f \in C(T).$$

But $\|U\| = o\text{-}\operatorname*{var}_{t\in T} g(t) \le C$. Clearly $\hat{U}(j) = \hat{g}(j) = y_j$, $j \in \mathbb{Z}$. □

It is useful to establish the Parseval formula explicitly also for the Fourier–Stieltjes series of the function g of o-bounded variation.

THEOREM 6. *Let Y be a complete vector lattice and let $f \in C(T)$. Then we have*

$$\int_T f(t)\,dg(t) = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|y|}{N+1}\right) \hat{f}(j)\,\hat{g}(-j).$$

P r o o f . By the Parseval formula from Theorem 1 the last equality holds for $f \in C(\mathbf{T})$. But since $C(\mathbf{T})$ is dense in $C(T)$ by the Stone-Weierstrass theorem we obtain the required result. □

It is a very important fact that we have established not only a characterization of the Fourier–Stieltjes (Fourier) series of the function of o-bounded variation (of the o-bounded linear mapping on $C(T)$ into Y) but also a method how to recapture the function (the mapping) by means of its Fourier–Stieltjes (Fourier) series. Theorem 6 (Theorem 1) gives a recipe how to recover the function (the mapping). In this sense we may, by abuse of notation, write

$$\mathrm{d}\boldsymbol{g}(t) \sim \sum_{j\in\mathbb{Z}} \hat{\boldsymbol{g}}(j)e^{ijx}$$

for $\boldsymbol{g} \in BV^o(T,Y)$.

It is easy to see that if the function $\boldsymbol{g} : T \to Y$ is nondecreasing then $\boldsymbol{g}$ is of o-bounded variation. Hence we may establish the following.

THEOREM 7. *Let Y be a complete vector lattice. The necessary and sufficient condition for*

$$\sum_{k\in\mathbb{Z}} y_k e^{ikx}$$

to be the Fourier–Stieltjes series of nondecreasing function $\boldsymbol{g}$ with the values in Y is that $\sigma_N(Y,t) \geq 0$ for all N in T.

P r o o f . The necessity. If $y_k = \hat{\boldsymbol{g}}(k)$ for a nondecreasing function $\boldsymbol{g}$ and if $f \in C(T)$ is nonnegative we have

$$\frac{1}{2\pi}\int_T f(t)\sigma_N(Y,t)\,\mathrm{d}t = \sum_{-N}^{N}\left(1-\frac{|y|}{N+1}\right)\hat{f}(j)\,\hat{\boldsymbol{g}}(-j) = \int_T \sigma_n(f,t)\,\mathrm{d}\boldsymbol{g}(t) \geq 0\,,$$

since $\boldsymbol{g}$ is nondecreasing and $\sigma_n(f,t) \geq 0$. Since this is true for arbitrary nonnegative f, we have $\sigma_N(Y,t) \geq 0$ on T.

Assuming $\sigma_N(Y,t) \geq 0$ we obtain

$$\|S_N(Y)\| = \sup_{\|f\|\leq 1}\left|\int_T f(t)\sigma_N(Y,t)\,\mathrm{d}t\right| = \frac{1}{2\pi}\int_T \sigma_N(Y,t)\,\mathrm{d}t = y_0\,,$$

and by Theorem 5

$$\sum_{j\in\mathbb{Z}} y_j e^{ijx}$$

is the Fourier–Stieltjes series for some $\boldsymbol{g} \in BV^0(T,Y)$. For arbitrary nonnegative $f \in C(T)$

$$\int_T f(t)\,\mathrm{d}\boldsymbol{g}(t) = \lim_{N\to\infty} \frac{1}{2\pi} \int_T f(t)\sigma_N(Y,t)\,\mathrm{d}t \geq 0\,,$$

and it follows that $\boldsymbol{g}$ is nondecreasing. □

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