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FUNCTIONS WITH A CLOSED GRAPH AND BILATERAL QUASICONTINUITY

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. In the paper the relationship between bilateral quasicontinuity and closedness of graph of functions is investigated. Moreover, a characterization of the set of points of discontinuity of quasicontinuous functions with closed graphs is given.

There are many papers which deal with the closed graph functions. (See for example [1], [2], and [4–6].) In the paper [2] the quasicontinuity of the composite functions of the form $g(f)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary closed graph function and $g\colon \mathbb{R} \to \mathbb{R}$ is a suitable continuous function is studied. The purpose of this note is to extend some results of [1] and [2].

We say that a function f from a space X into a space Y *has a closed graph* if the graph of the function f, i.e., $\{(x,y) \in X \times Y;\, y = f(x)\}$ is a closed subset of the product $X \times Y$. We denote by $C_f(D_f)$ the set of all points at which the function f is continuous (discontinuous).

The following result can be established by using a method similar to the one used in establishing [2; Theorem 1]. The symbols $L^-(f,a)$, $L^+(f,a)$ denote the cluster sets from the left and right, respectively, of the function $f\colon \mathbb{R} \to \mathbb{R}$ at the point a.

PROPOSITION 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ have a closed graph. Let $a \in \mathbb{R}$ be such that $L^-(f,a) \cap \{-\infty,+\infty\} \neq \emptyset$ $(L^+(f,a) \cap \{-\infty,+\infty\} \neq \emptyset)$. Then for each $\varepsilon > 0$ there is an interval $J \subset (a-\varepsilon, a) \cap C_f$ $(J \subset (a, a+\varepsilon) \cap C_f)$ such that f is unbounded on J.*

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PROPOSITION 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ have a closed graph. Let $a \in \mathbb{R}$ be such that $L^-(f, a) \cap \{-\infty, +\infty\} = \emptyset$ $(L^+(f, a) \cap \{-\infty, +\infty\} = \emptyset$. Then there is $\delta > 0$ such that f is bounded on $(a-\delta, a)$ (on $(a, a+\delta)$).*

P r o o f . Suppose that $a \in D_f$. (The opposite case is evident.) First we show that there is $\delta > 0$ such that $(a-2\delta, a) \subset C_f$. Suppose to the contrary that for each $n \in \mathbb{N}$ we have $D_f \cap (a-n^{-1}, a) \neq \emptyset$. Let $n \in \mathbb{N}$. Then by [2; Theorem 1] there is an interval $J_n \subset (a-n^{-1}, a) \cap C_f$ such that f is unbounded on J_n. Choose $x_n \in J_n$ such that $|f(x_n)| > n$. Then $L^-(f, a) \cap \{-\infty, +\infty\} \neq \emptyset$, which contradicts the assumption.

Now suppose to the contrary that f is unbounded on $(a-\delta, a)$. Let $n \in \mathbb{N}$ be such that $n^{-1} < \delta$. Since f is bounded on $[a-\delta, a-n^{-1}]$, there is $x_n \in (a-n^{-1}, a)$ such that $|f(x_n)| > n$. Then $L^-(f, a) \cap \{-\infty + \infty\} \neq \emptyset$ which contradicts the assumption.

(The second part of the proof is similar.) □

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *left* (*right*) *hand sided quasicontinuous at a point* $a \in \mathbb{R}$ if for every $\varepsilon > 0$ and for every neighbourhood V of $f(a)$ there exists a nonempty open set $W \subset (a-\varepsilon, a) \cap f^{-1}(V)$ $(W \subset (a, a+\varepsilon) \cap f^{-1}(V))$. f is *quasicontinuous* (*bilaterally quasicontinuous*) *at* a if it is both left or (and) right sided quasicontinuous at this point. (See [3].)

According to the previous Propositions we obtain the following result, which is an improvement of [2; Theorem 3]. (The proof is similar to the one used in establishing [2; Theorem 3].)

THEOREM 1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous. Then the following statements are equivalent:*

(i) *for each closed graph function $f\colon \mathbb{R} \to \mathbb{R}$ the composite function $g(f)$ is bilaterally quasicontinuous,*

(ii) *for each open set V in $\mathbb{R}$ such that $g^{-1}(V) \neq \emptyset$, $\sup g^{-1}(V) = +\infty$ and $\inf g^{-1}(V) = -\infty$.*

THEOREM 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bilaterally quasicontinuous function with a closed graph. Then f is continuous.*

P r o o f . By contradiction. Suppose that there is $a \in \mathbb{R}$ such that $L^+(f, a) \cap \{-\infty, +\infty\} \neq \emptyset$. (The case $L^-(f, a) \cap \{-\infty, +\infty\} \neq \emptyset$ is similar.) Let $\varepsilon > 0$ be arbitrary. Put $A = f^{-1}([f(a)-\varepsilon, f(a)+\varepsilon])$. Since f has a closed graph, the set A is closed in $\mathbb{R}$. Then there is a countable family $\mathcal{J}$ of pairwise disjoint open intervals such that $(\mathbb{R} - A) \cap (a, +\infty) = \cup\mathcal{J}$. Since f is right hand sided quasicontinuous at the point a, and $L^+(f, a) \cap \{-\infty, +\infty\} \neq \emptyset$, there is a sequence $\{J_n\}_{n=1}^{+\infty}$ such that $J_n \in \mathcal{J}$, and $a_n \to a$, where $a_n = \inf J_n$. Let

$n \in \mathbb{N}$. Since f is right sided quasicontinuous at the point a_n, we obtain $|f(a_n) - f(a)| = \varepsilon$. Thus $L^+(f, a) \cap \{f(a) - \varepsilon, f(a) + \varepsilon\} \neq \emptyset$, which contradicts the assumption. □

THEOREM 3. *Let $F \subset \mathbb{R}$. Then F is closed and nowhere dense if and only if there is a quasicontinuous function $f\colon \mathbb{R} \to \mathbb{R}$ with a closed graph such that $D_f = F$.*

Proof. $\Rightarrow$: Since $\mathbb{R} - F^d$ is open (where F^d is the set of all accumulation points of F), there is a countable family $\mathcal{J}$ of pairwise disjoint open intervals such that $\mathbb{R} - F^d = \cup\mathcal{J}$. Let $\mathcal{J}_1, \mathcal{J}_2$ be subfamilies of $\mathcal{J}$ such that the sets $\cup\mathcal{J}_1$, $\cup\mathcal{J}_2$ are disjoint and dense in F^d. Put $E = \mathbb{R} - \cup\mathcal{J}_1$. Define $g\colon \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} \dfrac{1}{\operatorname{dist}(x, E)}, & \text{if } x \notin E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a \in F$ be an isolated point of F. Then there is $\delta_a > 0$ such that $(a, a + 2 \cdot \delta_a) \cap F = \emptyset$. Put $I_a = (a, a + \delta_a)$. Define $h\colon \mathbb{R} \to \mathbb{R}$ as follows

$$h(x) = \begin{cases} \dfrac{\delta_a}{x - a} - 1, & \text{if } x \in I_a \text{ (where } a \in F - F^d\text{)}, \\ 0, & \text{otherwise.} \end{cases}$$

Put $f = g + h$. It is not difficult to verify that f is bilaterally quasicontinuous, f has a closed graph, and $D_f = F$.

$\Leftarrow$: By [1; Theorem 3]. □

REFERENCES

[1] BAGGS, I.: *Functions with a closed graph*, Proc. Amer. Math. Soc. **43** (1974), 439–442.

[2] DOBOŠ, J.: *On discontinuity points for closed graph functions*, Real Anal. Exchange **15** (1989–90), 337–339.

[3] GRANDE, J.—NATKANIEC, R.: *On quasi-continuous bijections*, Acta Math. Univ. Comenian. LX **1** (1991), 31–34.

[4] KOSTYRKO, P.—ŠALÁT, T.: *On functions whose graphs are closed sets* (*in Russian*), Čas. pěst. mat. **89** (1964), 426–432.

[5] KOSTYRKO, P.—NEUBRUNN, T.—ŠALÁT, T.: *On functions whose graphs are closed sets II* (*in Russian*), Acta F.R.N. Univ. Comenian. X, 3, Math. **12** (1965), 51–61.

[6] NEUBRUNN, T.: *c-continuity and closed graphs*, Čas. pěst. mat. **10** (1965), 172–178.

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