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ON A LATTICE STRUCTURE OF OPERATOR SPACES IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. For X, Y complete bornological locally convex spaces, we consider a lattice structure of the space $L(X, Y)$ of all continuous linear operators $L: X \to Y$.

Introduction

The description of theory of *complete bornological locally convex spaces* (C.B.L.C.S.) we can find in [4], [6], and [3].

In [1], [2] we have developed a technique for an operator valued measure $m: \Delta \to L(X, Y)$, where Δ is a δ-ring of sets, $L(X, Y)$ the space of all continuous operators $L: X \to Y$, where X, Y are both C.B.L.C.S. In [1] we gave a more detail explanation of basic $L(X, Y)$-measure set structures (H. Weber, cf. [7], considered these structures particularly from topological aspects). In connection with it, a Bartle type integral was investigated. In [2], convergences in measure, almost everywhere, almost uniform (and relations between them) were studied.

In the present paper we consider the lattice structure of the range space of such measure m, the space $L(X, Y)$.

1. Preliminaries

Let X, Y be two C.B.L.C.S. over the field of real or complex numbers

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equipped with the bornologies $\mathscr{B}_X, \mathscr{B}_Y$. The basis $\mathcal{U}$ of the bornology $\mathscr{B}_X$ has *a marked element* $u_0 \in \mathcal{U}$, if $u_0 \subset u$ for every $u \in \mathcal{U}$. Let the bases $\mathcal{U}, \mathcal{W}$ be chosen to consist of all $\mathscr{B}_X$–, $\mathscr{B}_Y$–bounded Banach disks in X, Y, with marked elements $u_0 \in \mathcal{U}$, $u_0 \neq \{0\}$, and $w_0 \in \mathcal{W}$, $w_0 \neq \{0\}$, respectively. Remind that *a Banach disk* in X is a set which is closed, absolutely convex and the linear span of which is a Banach space. The space X is an inductive limit of Banach spaces X_u, $u \in \mathcal{U}$,

$$X = \liminf_{u \in \mathcal{U}} X_u ,$$

cf. [4], where X_u is a linear span of $u \in \mathcal{U}$ and $\mathcal{U}$ is directed by inclusion (analogously for Y and $\mathcal{W}$).

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $u_1, u_2 \in \mathcal{U}$ we have: $u_1 \wedge u_2 = u_1 \cap u_2$, $u_1 \vee u_2 = \operatorname{acs}(u_1 \cup a_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, we write $(u_1, w_1) \ll (u_2, w_2)$ if and only if $u_1 \subset u_2$ and $w_1 \supset w_2$.

2. Lattice structure of $L(X, Y)$

If p_w is Minkowski functional of the set $w \in \mathcal{W}$, then for $u \in \mathcal{U}$, $L \in L(X,Y)$, we put $p_{u,w}(L) = \sup_{x \in u} p_w\big(L(x)\big)$. (If w does not absorb $L(x)$, $x \in u$, we put $p_{u,w}(L) = \infty$.) Denote by $\mathcal{L}_{u,w} = \{L \in L(X,Y) \colon p_{u,w}(L) < \infty\}$, $(u,w) \in \mathcal{U} \times \mathcal{W}$, and $\mathcal{L}_{\mathcal{U},\mathcal{W}} = \{\mathcal{L}_{u,w} \colon (u,w) \in \mathcal{U} \times \mathcal{W}\}$. For $(u,w) \in \mathcal{U} \times \mathcal{W}$, a sequence $L_n \in L(X,Y)$, $n = 1, 2, \dots$, is said to be convergent to $L \in L(X,Y)$ in $\mathcal{L}_{u,w}$ whenever $\lim_{n \to \infty} p_{u,w}(L_n - L) = 0$.

On $\mathcal{L}_{\mathcal{U},\mathcal{W}}$ define the operations $\wedge$, $\vee$ and *an order* $\ll$. For (u_1, w_1), $(u_2, w_2) \in \mathcal{U} \times \mathcal{W}$,

$$\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \wedge u_2, w_1 \vee w_2}, \mathcal{L}_{u_1,w_1} \wedge \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \vee u_2, w_1 \wedge w_2},$$
$$\mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u_1,w_1} \quad \text{if and only if} \quad (u_1, w_1) \ll (u_2, w_2).$$

It is easy to see that $\wedge$, $\vee$ are *lattice operations.*

THEOREM 1. *The family $\mathcal{L}_{\mathcal{U},\mathcal{W}}$ of operator spaces is a distributive lattice.*

Proof. For (u_1, w_1), (u_2, w_2), $(u_3, w_3) \in \mathcal{U} \times \mathcal{W}$, we have:

$$\begin{aligned}\mathcal{L}_{u_1,w_1} \vee (\mathcal{L}_{u_2,w_2} \wedge \mathcal{L}_{u_3,w_3}) &= \mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2 \vee u_3, w_2 \wedge w_3} \\ &= \mathcal{L}_{u_1 \wedge (u_2 \vee u_3), w_1 \vee (w_2 \wedge w_3)} \\ &= \mathcal{L}_{(u_1 \wedge u_2) \vee (u_1 \wedge u_3), (w_1 \vee w_2) \wedge (w_1 \vee w_3)} \\ &= \mathcal{L}_{u_1 \wedge u_2, w_1 \vee w_2} \wedge \mathcal{L}_{u_1 \wedge u_3, w_1 \vee w_2} \\ &= (\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2,w_2}) \wedge (\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_3,w_2}).\end{aligned}$$

By [5], Th. 2.2, the family $\mathcal{L}_{\mathcal{U},\mathcal{W}}$ is a distributive lattice. □

The lattice $\mathcal{L}_{\mathcal{U},\mathcal{W}}$ introduces a topology of an inductive limit on $L(\boldsymbol{X}, \boldsymbol{Y})$, i.e., there holds the following theorem.

THEOREM 2. $L(\boldsymbol{X}, \boldsymbol{Y}) = \liminf\limits_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w}$.

Proof. For $u \in \mathcal{U}$, $w \in \mathcal{W}$, it is easy to verify that $\mathcal{L}_{u,w}$ is a vector subspace of $L(\boldsymbol{X}, \boldsymbol{Y})$ equipped with the topology given by the seminorm $p_{u,w}$.

Show that $\bigcup\limits_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} = L(\boldsymbol{X}, \boldsymbol{Y})$. The inclusion $\bigcup\limits_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} \subset L(\boldsymbol{X}, \boldsymbol{Y})$ is trivial. Show $\bigcup\limits_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} \supset L(\boldsymbol{X}, \boldsymbol{Y})$. Let $L \in L(\boldsymbol{X}, \boldsymbol{Y})$. So, to each $u \in \mathcal{U}$ there exists $w_{u,L} \in \mathcal{W}$ such that $L(u) \subset w_{u,L}$, i.e., $p_{u,w_{u,L}}(L) \leq 1 < \infty$. Thus, $L \in \mathcal{L}_{u,w_{u,L}} \subset \bigcup\limits_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w}$.

Let (u_1, w_1), $(u_2, w_2) \in \mathcal{U} \times \mathcal{W}$. Show now that if $\mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u_1,w_1}$, then $\mathcal{L}_{u_2,w_2} \subset \mathcal{L}_{u_1,w_1}$ and if a sequence $L_n \in L(\boldsymbol{X}, \boldsymbol{Y})$, $n = 1, 2, \dots$, of operators converges to $L \in L(\boldsymbol{X}, \boldsymbol{Y})$ in $\mathcal{L}_{u_2,w_2}$, then it converges to L also in $\mathcal{L}_{u_1,w_1}$. Indeed, by definition, $(u_1, w_1) \ll (u_2, w_2) \iff u_1 \subset u_2$ and $w_1 \supset w_2$. The relation $u_1 \subset u_2$ implies $p_{u_1,w}(L) \leq p_{u_2,w}(L)$ for every $w \in \mathcal{W}$. The inclusion $w_2 \subset w_1$ implies $p_{w_1}(L(x)) \leq p_{w_2}(L(x))$ for every $x \in \boldsymbol{X}$. From this we have $p_{u,w_1}(L) \leq p_{u,w_2}(L)$ for every $u \in \mathcal{U}$. Thus, $p_{u_1,w_1}(L) \leq p_{u_1,w_2}(L) \leq p_{u_2,w_2}(L)$. So, if $(u_1, w_1) \ll (u_2, w_2)$ and $L \in L(\boldsymbol{X}, \boldsymbol{Y})$, then $p_{u_1,w_1}(L) \leq p_{u_2,w_2}(L)$. This completes the proof. □

Note that in the terminology of [6], $L(\boldsymbol{X}, \boldsymbol{Y})$ (as an inductive limit of seminormed spaces) is *a bornological convex vector space*, cf. [6], Chap. 4, §2, Th. 1.

THEOREM 3. *For every* $(u_1, w_1) \in \mathcal{U} \times \mathcal{W}$, *the set*

$$\mathcal{I}_{u_1,w_1} = \{\mathcal{L}_{u,w} \in \mathcal{L}_{\mathcal{U},\mathcal{W}}; \mathcal{L}_{u,w} \ll \mathcal{L}_{u_1,w_1}, (u,w) \in \mathcal{U} \times \mathcal{W}\}$$

is an ideal in $\mathcal{L}_{\mathcal{U},\mathcal{W}}$.

P r o o f . Let (p,q), $(u,w) \in \mathcal{U} \times \mathcal{W}$ and $(u_1,w_1) \ll (u,w)$, $(u_1,w_1) \ll (p,q)$. Since $u \wedge p = u \cap p \supset u_1$, $w \vee q = \operatorname{acs}(w \cup q) \subset w_1$, then $\mathcal{L}_{u,w} \vee \mathcal{L}_{p,q} = \mathcal{L}_{u\wedge p, w\vee q} \ll \mathcal{L}_{u_1,w_1}$.

Let (p,q), $(u,w) \in \mathcal{U} \times \mathcal{W}$, and $(u_1,w_1) \ll (p,q)$. Then $\mathcal{L}_{u,w} \wedge \mathcal{L}_{p,q} = \mathcal{L}_{u\vee p, w\wedge q} \ll \mathcal{L}_{u_1,w_1}$. □

Dually to Theorem 3, we obtain the following corollary.

COROLLARY 4. *For every* $(u_2,w_2) \in \mathcal{U} \times \mathcal{W}$, *the set*

$$\mathcal{F}_{u_2,w_2} = \{\mathcal{L}_{u,w} \in \mathcal{L}_{\mathcal{U},\mathcal{W}}; \mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u,w}, (u,w) \in \mathcal{U} \times \mathcal{W}\},$$

is a filter in $\mathcal{L}_{\mathcal{U},\mathcal{W}}$.

THEOREM 5. *Let* (u_1,w_1), $(u_2,w_2) \in \mathcal{U}\times\mathcal{W}$. *If* $(u_1,w_1) \ll (u_2,w_2)$, *then the order interval* $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}] = \mathcal{I}_{u_1,w_1} \cap \mathcal{F}_{u_2,w_2}$ *in* $\mathcal{L}_{\mathcal{U},\mathcal{W}}$ *is a Boolean algebra with* $\mathcal{L}_{u_2,w_2}$ *as null and* $\mathcal{L}_{u_1,w_1}$ *as unit.*

P r o o f . Let $(u,w) \in \mathcal{U} \times \mathcal{W}$, $(u_1,w_1) \ll (u,w) \ll (u_2,w_2)$. Put

$$\mathcal{L}^{\perp}_{u,w} = \mathcal{L}_{(u_2\setminus u)\vee u_1, (w_1\setminus w)\vee w_2} \in [\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$$

and show that $\mathcal{L}^{\perp}_{u,w}$ is a complement of $\mathcal{L}_{u,w}$ in $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$. We have:

$$\begin{aligned} \mathcal{L}_{u,w} \vee \mathcal{L}^{\perp}_{u,w} &= \mathcal{L}_{u,w} \vee \mathcal{L}_{(u_2\setminus u)\vee u_1, (w_1\setminus w)\vee w_2} \\ &= \mathcal{L}_{u\wedge[(u_2\setminus u)\vee u_1], w\vee[(w_1\setminus w)\vee w_2]} \\ &= \mathcal{L}_{[u\wedge(u_2\setminus u)]\vee[u\wedge u_1], w_1\vee w_2} \\ &= \mathcal{L}_{u_1,w_1}. \end{aligned}$$

Analogously, $\mathcal{L}_{u,w} \wedge \mathcal{L}^{\perp}_{u,w} = \mathcal{L}_{u_2,w_2}$. So, $\mathcal{L}_{u_2,w_2}$ is the null and $\mathcal{L}_{u_1,w_1}$ is the unit of the Boolean algebra $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$. □

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