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THE DARBOUX PROPERTY IN SOME FAMILIES OF BAIRE 1 FUNCTIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Denote by D the family of Darboux functions, by P the family of Peek's functions ([10]) pointwise discontinuous on each union of sequence of perfect sets, by G_1 the family of functions pointwise discontinuous on each non-empty set, and by Q the family of quasicontinuous functions. I investigate the addition and the multiplication in P and DP. Moreover, I show that $DP \subset Q$, and that $f \in G_1$ iff for every perfect set E there is an open interval I such that $I \cap E \neq \emptyset$ and $f|(I \cap E)$ is continuous.

Let $\mathbb{R}$ denote the set of all reals. For a given family K of functions $f\colon \mathbb{R} \to \mathbb{R}$ define

$$K + K = \{f + g\colon f, g \in K\}, \qquad K \cdot K = \{fg\colon f, g \in K\},$$

$$B_1(K) = \{f\colon \text{ there is a sequence of functions } f_n \in K \text{ such that } \lim_{n\to\infty} f_n = f\},$$

$$B_u(K) = \{f\colon \text{ there is a sequence of functions } f_n \in K \text{ which uniformly converges to } f\},$$

$$M_a(K) = \{f\colon \text{ for each } g \in K,\ f + g \in K\},$$

$$M_m(K) = \{f\colon \text{ for each } g \in K,\ fg \in K\}.$$

Moreover, let us put:

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$C = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous }\}$,

$D = \{f : \mathbb{R} \to \mathbb{R} : f \text{ has the Darboux property }\}$,

$P = \{f : \mathbb{R} \to \mathbb{R}$: for each sequence of nonempty perfect sets P_n, $n = 1, 2, \ldots$, the restricted function $f|\bigcup_n P_n$ is continuous at some point $x \in \bigcup_n P_n\}$ ([10]),

$G_1 = \{f : \mathbb{R} \to \mathbb{R}$: for every nonempty set $E \subset \mathbb{R}$ the restricted function $f|E$ is continuous at some point $x \in E\}$ ([5]),

$G_2 = \{f : \mathbb{R} \to \mathbb{R}$: for every nonempty countable set $E \subset \mathbb{R}$ the restricted function $f|E$ is continuous at some point $x \in E\}$ ([5]),

$B_1^* = \{f : \mathbb{R} \to \mathbb{R}$: for every nonempty perfect set $E \subset \mathbb{R}$ there is an open interval I such that $I \cap E \neq \emptyset$ and $f|(I \cap E)$ is continuous $\}$ ([3]),

$S = \{f : \mathbb{R} \to \mathbb{R}$: the set $C(f)$ of all continuity points of f is dense $\}$

$Q = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is quasicontinuous at each point } x \in \mathbb{R}\}$,

(f is quasicontinuous at x if for every $r > 0$ there is a nonempty open set $U \subset (x - r, x + r)$ with $f(U) \subset (f(x) - r, f(x) + r)$ ([8])),

$F(K) = \{f \in K$: if f is discontinuous from the right (resp. left) at x, then $f(x) = 0$ and there is a sequence $x_n \searrow x$ ($y_n \nearrow x$) such that $f(x_n) = 0$ $(f(y_n) = 0)\}$ ([4]),

$E(K) = \{f \in K$: f has a zero in each subinterval in which it changes sign$\}$ ([2]).

It is known that $P \subset B_1(C)$ ([10]), $G_1 = G_2 \not\subseteq P$ ([5]), $G_1 + G_1 = G_1$, $G_1 \cdot G_1 = G_1$, $DB_1^* + DB_1^* = B_1^*$ ([6]), $DB_1^* \cdot DB_1^* = E(B_1^*)$ ([6]), $M_a(DB_1^*) = C$, and $M_m(DB_1^*) = F(DB_1^*)$ ([6]). In this paper I prove that $G_1 = B_1^*$, $B_1(DG_2) = SB_1(B_1(C))$, $B_u(DG_2) = DQB_1(C)$, $P + P = P$, $P \cdot P = P$, $DP \subset Q$, $B_1(DP) = SB_1(B_1(C))$, $M_a(DP) = C$, and $M_m(DP) = F(DP)$.

THEOREM 1. $G_1 = G_2 = B_1^*$.

P r o o f. Let $f \in B_1^*$ and let $E \subset \mathbb{R}$ be a nonempty set. If there is an isolated point $x \in E$, then $f|E$ is continuous at x. In the contrary case the closure $\operatorname{cl} E$ of the set E is a perfect set and there is an open interval I such that $I \cap \operatorname{cl} E \neq \emptyset$ and $f|(I \cap \operatorname{cl} E)$ is continuous. So $f|E$ is continuous at each point $x \in I \cap E$ and $B_1^* \subset G_1 = G_2$. Now, let $f \in G_2$ and let $E \subset \mathbb{R}$ be a perfect set. If for every open interval I, with $I \cap E \neq \emptyset$, the restriction $f|(I \cap E)$ is discontinuous at a point $x \in I \cap E$, then there is a sequence of points $x_n \in E$ such that $\operatorname{cl}(\{x_n : n = 1, 2, \dots\}) = E$ and $\operatorname{osc}(f|(C(f|E) \cup \{x_n\}))(x_n) > 0$, $n = 1, 2, \dots$. Let $B = \{x_n : n = 1, 2, \dots\}$. Since $f \in G_2$, there is a point $u \in B$ such that $f|B$ is continuous at u. Let $b = \operatorname{osc}(f|C(f|E))(u)$. Since $u \in B$, $b > 0$. There is an open interval $I \ni u$ such that $|f(t) - f(u)| < b/4$ for each $t \in I \cap B$. We consider two cases:

(1) There are points $v, w \in I \cap C(f|E)$ such that $|f(v) - f(w)| > 3b/4$. Then there are open intervals $I_1, I_2 \subset I$ such that $v \in I_1$, $w \in I_2$, $|f(t) - f(v)| < b/8$ for each $t \in E \cap I_1$, and $|f(t) - f(w)| < b/8$ for each $t \in E \cap I_2$. Let $s, z \in B$ such that $s \in I_1$, $z \in I_2$. We have

$$|f(v) - f(w)| = |(f(v) - f(s)) + (f(s) - f(z)) + (f(z) - f(w))| < \\ < b/8 + |f(s) - f(z)| + b/8,$$

and

$$|f(s) - f(z)| > |f(v) - f(w)| - b/4 > 3b/4 - b/4 = b/2,$$

in contradiction with

$$|f(s) - f(z)| \leq |f(s) - f(u)| + |f(z) - f(u)| < b/4 + b/4 = b/2.$$

(2) There is a point $w \in I \cap C(f|E)$ such that $|f(w) - f(u)| > 3b/4$. Let $J \ni w$ be an open interval such that $w \in J \subset I$ and $|f(t) - f(w)| < b/4$ for each $t \in E \cap J$. There is a point $v \in B \cap J$. We have

$$|f(v) - f(u)| > |f(w) - f(u)| - b/4 > 3b/4 - b/4 = b/2,$$

in contradiction with

$$|f(v) - f(u)| < b/4.$$

So there is an open interval I such that $E \cap I \neq \emptyset$ and $f|(E \cap I)$ is continuous. Consequently, $f \in B_1^*$ and the proof is completed. □

R e m a r k 1. $B_1(DG_2) = SB_1(B_1(C))$.

P r o o f. The proof is the same as the proof of Theorem 1 from [7]. □

THEOREM 2. $B_u(DG_2) = DQB_1(C)$.

Proof. Since $DB_1^* \subset Q$ ([6]) and $B_u(Q) \subset Q$ ([9]) and $B_u(DB_1(C)) = DB_1(C)$ ([1]), we have

$$B_u(DG_2) = B_u(DB_1^*) \subset DQB_1(C).$$

Let $f \in DQB_1(C)$ be a function. Fix $r > 0$. There is a step-like functions (i.e. for every nonempty set $E \subset \mathbb{R}$ there is an open interval I such that $I \cap E \neq \emptyset$ and $g|(I \cap E)$ is constant) such that $|f(x) - g(x)| < r/8$ for each $x \in \mathbb{R}$ ([10]). The image $g(\mathbb{R})$ is countable ([5]), so $g(\mathbb{R}) = \{y_1, y_2, \dots\}$. If (a, b) is the interior of some component of the set $g^{-1}(y_k)$, then there is a continuous function $g_{ab}\colon (a, b) \to [y_k - r/2,\ y_k + r/2]$ such that both cluster sets

$$K^+(g_{ab}, a) = \{y \in \mathbb{R}\colon \text{ there is a sequence } x_n \searrow a \text{ such that } g_{ab}(x_n) \to y\},$$

and

$$K^-(g_{ab}, b) = \{y \in \mathbb{R}\colon \text{ there is a sequence } x_n \nearrow b \text{ such that } g_{ab}(x_n) \to y\}$$

are equal to $[y_k - r/2,\ y_k + r/2]$. Let us put

$$h(x) = \begin{cases} g_{ab}(x) & \text{if } x \text{ belongs to the interior } (a,b) \text{ of some component} \\ & \text{of the level set } g^{-1}(y_k),\ k = 1, 2, \dots, \text{ and} \\ g(x) & \text{otherwise.} \end{cases}$$

Since $g \in B_1^*$, $h \in G_2$. Moreover

$$|h(x) - f(x)| \le |h(x) - g(x)| + |g(x) - f(x)| \le r/2 + r/8 < r$$

for each $x \in \mathbb{R}$. We shall show that h has the Darboux property. Fix $u \in \mathbb{R}$. Since h is of Baire 1, it suffices to show ([1]) that there are sequences $x_n \nearrow u$, $v_n \searrow u$ such that $h(x_n) \to h(u)$ and $h(v_n) \to h(u)$. We shall show only the existence of such sequence (x_n). If $u \in \operatorname{int} g^{-1}(y_k)$ (int denotes the interior) for some $k = 1, 2, \dots$, then every sequence $x_n \searrow u$ is such that $h(x_n) \to h(u)$.
If u is the left end of some component of the level set $g^{-1}(y_k)$, $k = 1, 2, \dots$, then there is a sequence (x_n) of points belonging to this component such that $x_n \searrow u$ and $h(x_n) \to h(u)$. Suppose that $u \in g^{-1}(y_k) - \operatorname{int} g^{-1}(y_k)$ for some index k and u is not the left end of a component of the level set $g^{-1}(y_k)$. Since $f \in DQB_1(C)$ and $|f(x) - g(x)| < r/8$ for each $x \in \mathbb{R}$, there is a sequence $I_m = (a_m, b_m)$ of the interiors of nondegenerate components of the level sets $g^{-1}(y_{k_m})$ such that $b_m \searrow u$ and $y_k \in (y_{k_m} - r/2,\ y_{k_m} + r/2)$ for $m = 1, 2, 3, \dots$. Really, the union U of all open intervals on which g is constant is dense. Thus, if such sequence (I_m) is not, then there is $r_1 > 0$ such that

(i) $|g(x) - y_k| \ge r/2$ for each $x \in (u, u + r_1) \cap U$.

But $|g(x)-g(u)| \le |f(x)-g(x)| + |f(x)-f(u)| + |g(u)-f(u)| <$
$< r/8 + |f(x)-f(u)| + r/8 = |f(x)-f(u)| + r/4$
for each $x \in \mathbb{R}$. Since $f \in DQ$, there is a point $w \in U \cap (u, u+r_1)$ such that $|f(w)-f(u)| < r/4$. Consequently,

$$|g(w)-y_k| = |g(w)-g(u)| \le |f(w)-f(u)| + r/4 < r/4 + r/4 = r/2,$$

in contradiction with (i). So, such sequence (I_m) must exist. In every interval I_m, $m = 1, 2, \dots$, there is a point x_m such that $h(x_m) = y_k$. Thus $h(x_m) \to y_k$ and the proof is completed. □

THEOREM 3. *Let $f, g \in P$ and let (A_n) be a sequence of perfect sets. There is a point $x \in A = \bigcup_n A_n$ at which both restricted functions $f|A$, $g|A$ are continuous.*

P r o o f. Suppose that such point x is not. Let

$$A_{nk} = \{t \in A_n \colon \operatorname{osc}(f|A)(t) \ge 1/k\},$$
$$B_{nk} = \{t \in A_{nk} \colon t \text{ is a point of condensation of } A_{nk}\},$$
$$C_{nk} = \{t \in A_n \colon \operatorname{osc}(g|A)(t) \ge 1/k\},$$
$$D_{nk} = \{t \in C_{nk} \colon t \text{ is a point of condensation of } C_{nk}\}.$$

We have,

$$A = \bigcup_{n,k=1}^{\infty} (A_{nk} \cup C_{nk}).$$

The set A is c-dense in itself and the set $B = A - \bigcup_{n,k=1}^{\infty} (B_{nk} \cup D_{nk})$ is countable. So, the set $A - B$ is c-dense in itself and dense in A. There is an open interval I such that $I \cap A \ne \emptyset$ and

$$I \cap \bigcup_{n,k} B_{nk}, \quad \text{or} \quad I \cap \bigcup_{n,k} D_{nk}$$

is dense in $I \cap A$. We may assume that $I \cap \bigcup_{n,k} B_{n,k}$ is dense in $I \cap A$. Since every set $B_{n,k} (n, k = 1, 2, \dots)$ is perfect (whenever $B_{nk} \ne \emptyset$), and $I \cap \bigcup_{n,k} B_{nk}$ is dense in $I \cap A$, the set

$$E = I \cap \bigcup_{n,k} A_{nk}$$

is the union of a sequence of perfect sets. Consequently, there is a point $v \in E$ at which the restricted function $f|E$ is continuous. Since $v \in E$, there are indices n_0, k_0 such that $v \in A_{n_0 k_0}$. Thus $\operatorname{osc}(f|A)(v) \geq 1/k_0 > 0$. Let $r = 1/8k_0$. There is an open interval $J \subset I$ such that $v \in J$ and $|f(t) - f(v)| < r$ for each point $t \in E \cap J$. Since $\operatorname{osc}(f|A)(v) \geq 1/k_0 = 8r$, there is a point $w \in J \cap A$ such that

(i) $|f(v) - f(w)| > 3r$.

Evidently, $w \in (J \cap A) - E$ is a continuity point of the restricted function $f|A$. Thus there is an open interval $K \subset J$ such that $w \in K$ and

$|f(w) - f(t)| < r$ for each point $t \in K \cap A$. Let $u \in K \cap E$. Then

$|f(u) - f(v)| < r, \quad |f(u) - f(w)| < r,$

and

$|f(w) - f(v)| \leq |f(w) - f(u)| + |f(u) - f(v)| < r + r = 2r,$

in contradiction with (i). This completes the proof. □

As an immediate corollary we obtain:

COROLLARY 1. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $f, g \in P$. Then the function $x \to F\big(f(x), g(x)\big)$ belongs to P. In particularity, $f+g \in P$, $f \cdot g \in P$, $\max(f,g) \in P$, and $\min(f,g) \in P$.*

THEOREM 4. *$DP \subset Q$.*

Proof. Suppose that there is $f \in DP - Q$. Denote by $G\big(f|C(f)\big)$ the graph of the restricted function $f|C(f)$ and remark that there is a point x such that $(x, f(x)) \notin \operatorname{cl}\big((G(f|C(f))\big)$. Let $r > 0$ be such that $[x - r, x + r] \times [f(x) - r, f(x) + r] \cap \operatorname{cl}\big(G\big(f|C(f)\big)\big) = \emptyset$. Since $f \in DP \subset DB_1(C)$, the set $A = f^{-1}\big((f(x) - r, f(x) + r)\big) \cap (x - r, x + r)$ is the union of a sequence of perfect sets. If there is an open interval I such that $I \cap A$ is dense in I then the restricted function $f|I$ is discontinuous at each point $t \in I$, in contradiction with $f \in B_1(C)$. Thus the set A is nowhere dense and consequently $\operatorname{cl} A$ is also nowhere dense. Since $f \in P$ and A is the union of a sequence of perfect sets, there is a point $u \in A$ at which the restricted function $f|A$ is continuous. Let $I \subset (x - r, x + r)$ be an open interval such that $u \in I$ and

(i) $|f(t) - f(x)| < r_1 < r$ for each $t \in A \cap I$.

Let (a, b) be a component of the set $\mathbb{R} - \operatorname{cl} A$ such that a (or b) belongs to $\operatorname{cl} A$. Since f has the Darboux property and

$|f(t) - f(x)| \geq r$ for each $t \in (a, b)$, we have

(ii) $|f(a)-f(x)| \geq r$.

From (i) and (ii) it follows that the restricted function $f|(I \cap \mathrm{cl}\,A)$ is discontinuous at each point $t \in I \cap \mathrm{cl}\,A$, in contradiction with $f \in B_1(C)$. This completes the proof. □

From Theorem 4 and Remark 1 it follows:

THEOREM 5. $B_1(DP) = SB_1(B_1(C))$.

From Theorem 4 and Theorem 2 it follows:

THEOREM 6. $B_u(DP) = DQB_1(C)$.

THEOREM 7. $M_a(DP) = C$.

The proof of this theorem is the same as the proof of Bruckner's theorem 3.2 in [1].

THEOREM 8. $M_m(DP) = F(DP)$.

Proof. The proof is a simple modification of the proof of Fleissner's theorem from [4]. The proof of the inclusion $F(D) \subset M_m(DP)$ follows from Corollary 1 and Fleissner's theorem from [4]. For the proof of the inclusion $M_m(DP) \subset F(DP)$ we suppose that there is a function $f \in M_m(DP)-F(DP)$. The same as in Fleissner's proof from [4] we can limit the considerations to two cases.

Case 1. Suppose that f is discontinuous from the right at a and $f(x)>0$ on $(a,a+r]\,(r>0)$. Choose $K>0$ such that there is a sequence $p_n \searrow a$ where $\lim_{n\to\infty} f(p_n) = K \neq f(a)$. Set

$$g(x) = \begin{cases} 1/f(a+r) & \text{for } x \geq a+r, \\ 1/f(x) & \text{for } x \in (a,\, a+r), \\ 1/K & \text{for } x \leq a, \end{cases}$$

Then $g \in DP$, but $f(a)g(a) \neq 1$ and $f(x)g(x) = 1$ on $(a,a+r)$. So, fg has not the Darboux property.

Case 2. Suppose that f is discontinuous from the right at a, $f(a)>0$ and there exists a sequence $p_n \searrow a$ such that $f(p_n) = 0$ for $n = 1,2,\dots$. Set $E = \{x > a\colon f(x) < f(a)/2\}$. Since $f \in DP \subset Q$ and $p_n \in E$, $n = 1,2,\dots$, there are disjoint closed intervals $I_n = [a_n,\, b_n]$, $n = 1,2,\dots$, contained in E and such that $a_{n+1} < b_{n+1} < a_n$, $n = 1,2,\dots$, $a_n \to a$, $b_n \to a$. Consequently, there is a function $g \in DP$ such that $0 < g(x) \leq 1$ for $x \in \bigcup_n I_n$, $g(x) = 0$ for $x \in (a, \infty) - \bigcup_n I_n$, and $g(x) = 1$ for $x \leq a$. Then $g(a)f(a) = f(a)$ and $f(x)g(x) < f(a)/2$ for $x > a$. Thus fg has not the Darboux property. □

R e m a r k 2. Well know that $DB_1(C) \cdot DB_1(C) = E(B_1(C))$ ([2]), and $DB_1^* \cdot DB_1^* = E(B_1^*)$ ([6]). Remark that $DP \cdot DP \neq E(P)$. For example, if a sequence (x_n) is dense in $\mathbb{R}$ and

$$f(x) = \begin{cases} 1/n & \text{if } x = x_n, \quad n = 1, 2, \dots, \\ 0 & \text{otherwise}, \end{cases}$$

then $f \in E(P)$, but f is not the product of a finite family of quasicontinuous functions.

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