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TWO THEOREMS ON VECTOR LATTICE–VALUED RANDOM VARIABLES

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. The aim of this paper is to investigate random variables taking on values in vector lattices. In the first part a result of Chow and Lai [2] on weighted sums of real valued random variables is extended to vector lattices. In the second part an ergodic theorem for vector lattice-valued random variables is proved.

I.

The terminology follows [3] and [5].

Definition 1. Let (Z, S, P) be a probability space, E a vector lattice. A sequence (f_n) of functions from Z to E converges to a function $f\colon Z \to E$ *almost uniformly* if for every $\varepsilon > 0$ there exist a set $A \in S$ such that $P(A) < \varepsilon$, a sequence (a_n) of real numbers converging to zero and an element $r \in E$ such that $|f_n(z) - f(z)| \le a_n r$ for each $z \in Z - A$.

Definition 2. A function $f\colon Z \to E$ is called a *random variable* if there exists a sequence (f_n) of simple E-valued functions such that (f_n) converges to f almost uniformly.

In what follows the notion of the σ-complete vector lattice with the σ-property as well as that of F–lattice are needed.

Definition 3. A vector lattice E is said to be *σ-complete* if every non-empty at most countable subset of E which is bounded from above has a supremum. E is said to have the *σ-property* if every countable set in E is included in a principal ideal of E (cf. [5]).

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DEFINITION 4. An Archimedean vector lattice E with a monotonous F–norm (not-necessarily homogeneous) complete with respect to it is called a *Fréchet lattice* (*F–lattice* for short) (cf. [9]).

In several former papers we studied the order-convergence of weighted sums of F–lattice valued random variables. The novelty of this paper is that the weights are allowed to take on their values randomly.

PROPOSITION 1. *Let E be an F–lattice, P a complete probability measure. Then each random variable is a random element in the sense of* [6], *i.e., a measurable map from Z to E.*

See [8] for the proof.

THEOREM 1. *Let E be a σ-complete F–lattice with the σ-property. If f_n are independent, identically distributed, symmetric random variables in E such that*

$$\sum_{n=1}^{\infty} P\{z;\ |f_1(z)| \le n\,a\}^C < \infty \quad \text{for some } a \in E\,,\ a > 0\,,$$

and a_{nk} are real random variables such that

$$P\{\limsup \sum_{k=1}^{n} a_{nk}^2 \le G\} = 1 \quad \text{for some constant } G\,,\ \text{then}$$

$$n^{-1} \sum_{k=1}^{n} a_{nk} f_k \to 0 \quad \text{with probability one.}$$

P r o o f . For each n let (f_{nk}) be a sequence of simple functions converging almost uniformly to f_n. It means that there are at most countable many different regulators of the order-convergence. Because of the assumption of the σ-property and because of the inequality

$$|f_n| \le |f_n - f_{nk}| + |f_{nk}|$$

we obtain that all values of f_n belong to an ideal generated by a single element, say $u, u \in E$. Let us denote this ideal by I_u, the set of all values which the variables f_{nk}, $n, k \in N$ take on by $(y_n)_{n=1}^{\infty}$ and put $y_0 = u$. Consider the countable set A of all linear combinations of the elements y_n with the rational coefficients. It is evident that the set

$$B = \bigcap_{r \in Q} \bigcup_{a \in A} \{x \in I_u;\ |x^{-a}| \le r\,u\}\,,$$

where Q stands for the set of all rationals is a linear subspace of I_u. Equipped with the order – unit norm inherited from I_u B, being a closed subset of I_u, becomes a separable Banach space. This space will be denoted by $(B, \| \, \|_u)$. Denote by W_s the Borel σ-algebra subsets of B generated by the open balls and by W_T the σ-algebra generated by the subsets of B which are open in the original topology. Because of the equality

$$\{x \in B;\ \|x - x_i\|_u < \varepsilon\} = \bigcup_n B \cap \{x \in I_u;\ \|x - x_i\|_u \leq \varepsilon(1 - n^{-1})\} =$$

$$= B \cap \bigcup_n \{x \in I_u;\ |x - x_i| \leq \varepsilon(1 - n^{-1})u\},$$

which holds for each open ball we have that $W_s \subset W_T$. It follows then that f_n can be regarded as independent, identically distributed, symmetric random variables in $(B, \| \, \|_u)$. Moreover we have

$$E\|f_1\|_u \leq 1 + \sum_{n=1}^{\infty} P\left(\|f_1\|_u > n\right) = 1 + \sum_{n=1}^{\infty} P\left(|f_1| \leq n\, u\right)^C < \infty .$$

The rest of the proof follows from [6], Theorem 6.1.2. □

II.

Ergodic theorems for vector-lattice valued random variables can be found, e.g., in [4]. They are proved, however, under stringent conditions on random variables. In our version random variables are allowed to be far more general. For terminology see [1].

DEFINITION 5. Let (Z, S, P) be a probability space, E a σ-complete vector lattice with the σ-property. A non-negative function $f \colon Z \to E$ is called an *integrable random variable* if there exists a non-decreasing sequence (f_n) of non-negative simple functions such that (f_n) converges to f almost uniformly and the sequence (Ef_n) of their expectations converges relatively uniformly. We define the integral (the expected value) of f by $Ef = \text{ru–lim}\, Ef_n$.

A function $f \colon Z \to E$ is said to be an *integrable random variable* if there exist non-negative integrable random variables f_1 and f_2 such that $f = f_1 - f_2$. The integral (the expected value) is defined by setting $Ef = Ef_1 - Ef_2$.

The correctness of this definition is proved in [7]. Moreover we showed that the following theorem holds.

THEOREM 2. *If (f_n) is a non-decreasing sequence of random variables with expected values Ef_n almost uniformly converging to a random variable f and* *such that* ru–lim Ef_n *exists, then* f *has the expected value* Ef *and* $Ef =$ ru–lim Ef_n .

We recall that a probability preserving transformation $T\colon Z \to Z$ is ergodic if $P(B) = 0$ or $P(B) = 1$ for each set $B \in S$ invariant under T.

THEOREM 3. *Let* (Z, S, P) *be a probability space, let* $T\colon Z \to Z$ *be a probability preserving transformation. Then for every integrable random variable* f *there exists an invariant integrable random variable* f^* *such that* $\lim \frac{1}{n} \sum\limits_0^{n-1} f \cdot T^i = f^*$ *almost surely with respect to the order and* $Ef^* = Ef$. *If* T *is an ergodic transformation, then* $\lim \frac{1}{n} \sum\limits_0^{n-1} f \cdot T^i = Ef$ *almost surely with respect to the order.*

P r o o f. Let $f = c\, I_B$, $c \in E$, $B \in S$. Then by [1] Th. 1.3 there exists an invariant set $A \in S$, $P(A^C) = 0$ and a bounded invariant integrable real random variable g such that $\lim \frac{1}{n} \sum\limits_0^{n-1} I_B(T^i z) = g(z)$ for each $z \in A$. It follows that $\lim \frac{1}{n} \sum\limits_0^{n-1} f(T^i z) = c\, g(z)$ with respect to the order for each $z \in A$. Moreover $c\,g$ is an invariant integrable random variable and $E(c\,g) = c\,E\,g = c\,E\,I_B = Ef$.

If T is an ergodic transformation, then $\lim \frac{1}{n} \sum\limits_0^{n-1} I_B(T^i z) = E\,I_B$ almost surely and hence $\lim \frac{1}{n} \sum\limits_0^{n-1} f(T^i z) = E\,f$ almost surely.

If f is a simple function, that is $f = \sum\limits_1^n a_i\, I_{B_i}$, $a_i \in E$, $B_i \in S$, $i = 1, \dots, n$, then by the first part of the proof there exist invariant integrable random variables f_i^*, $i = 1, \dots, n$ and invariant A_i, $i = 1, \dots, n$ such that $\lim \frac{1}{n} \sum\limits_0^{n-1} f_j(T^i z) = f_j(z)$ for each $z \in A_j$, where $f_j = a_j\, I_{B_j}$, $j = 1, \dots, n$. Moreover $Ef_j^* = Ef_j$. It follows that $f^* = \sum\limits_1^n f_j^*$ is an invariant integrable random variable and $A = \bigcap\limits_1^n A_j$ is an invariant set such that $P(A^C) = P \bigcup\limits_1^n A_j^C = 0$. We have $\lim \frac{1}{k} \sum\limits_0^{k-1} f(T^i z) = \lim \frac{1}{k} \sum\limits_0^{k-1} \sum\limits_{j=1}^{n} a_j\, I_{B_j}(T^i z) = \sum\limits_{j=1}^{n} \lim \frac{1}{k} \sum\limits_0^{k-1} f_j(T^i z) = f^*(z)$ for each $z \in A$ and $Ef^* = \sum\limits_1^n E\,f_j^* = \sum\limits_1^n E\,f_j = Ef$. The part of the theorem concerning the ergodic transformation can be proved analogically.

Let f be a non-negative integrable random variable. Then by Definition 5 there exists a non-decreasing sequence of simple functions almost uniformly converging to f. Omitting if necessary a set of probability 0, we have, by this definition, that there exists a countable partition (B_k) of Z such that $|f_n(z) - f(z)| \leq a_n^k r$, for each $z \in B_k$, $r \in E$, $a_n^k \downarrow^k 0$, $a_n^k \in \mathbb{R}$, $n, k = 1, 2, \dots$. The existence of a common regulator of the order-convergence r follows from the fact that E has the σ-property.

For each n there exists, by the previous part of the proof, an invariant integrable random variable f_n^* such that

$$\lim \frac{1}{k} \sum_0^{k-1} f_n(T^i z) = f_n^*(z) \quad \text{for each} \quad z \in A_n,\ A_n \in S,\ P(A_n^C) = 0 .$$

Since all A_n are invariant sets, the set $A = \bigcap_1^\infty A_n$ is also invariant and $P(A) = 1$. Denoting $\frac{1}{k} \sum_0^{k-1} f(T^i z)$ by $S_k(z)$, $k = 1, \dots$ we have $\left|\frac{1}{k} \sum_0^{k-1} f_n(T^i z) - S_k(z)\right| \leq a_n^j r$ on B_j and hence $f_n^*(z) - a_n^j r \leq \limsup S_k(z) \leq f_n^*(z) + a_n^j r$ for each $z \in A \cap B_j$, $j = 1, 2, \dots$. Since the similar inequality holds for $\liminf S_k(z)$, we obtain that $\lim \frac{1}{k} \sum_0^{k-1} f(T^i z)$ exists almost surely. Define the function f^* as follows: $f^*(z) = \lim \frac{1}{k} \sum_0^{k-1} f(T^i z)$ for $z \in A$ and $f^*(z) =$ otherwise, and put $g_n = f_n^* I_A$, $n = 1, 2, \dots$. Since A is an invariant set, g_n are invariant functions. Moreover since $f_n \leq f_{n+1}$ for each natural n, the sequence (g_n) is non-decreasing. It follows from the above inequality that (g_n) ru–converges. Hence, by Theorem 2, f^* is an integrable random variable such that $E f^* = \lim E g_n = \lim E f_n = E f$. Since $f^*(z) = \lim g_n(z) = \lim g_n(Tz) = f^*(Tz)$ we have that f^* is an invariant function.

If T is an ergodic transformation, $f_n^* = E f_n$ almost surely implies $g_n = E f_n$ almost surely and finally $f^* = \lim E f_n = E f$ almost surely.

If f is an arbitrary integrable random variable, then there exist non-negative integrable random variables f_1 and f_2 such that $f = f_1 - f_2$. This part of the proof is obvious and therefore will be omitted. □

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