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ON QUASIOSCILLATION

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Let X be a topological space and (Y,d) be a metric space. If $f\colon X \to Y$ is a function, then there is a function $k_f\colon X \to \mathbb{R} \cup \{\infty\}$ such that f is cliquish at x if and only if $k_f(x) = 0$. Some properties of this function are studied.

The present paper was motived by [7], where some properties of the oscillation are investigated. An analogical function resembling the oscillation characterizing the cliquishness points can be defined.

Let us recall some concepts.

Let X be a topological space and let (Y,d) be a metric space.

A function $f\colon X \to Y$ is said to be *cliquish at* $x \in X$ ([10]) if for each $\varepsilon > 0$ and each neighbourhood U of x there is an open nonempty set $G \subset U$ such that $d\big(f(y), f(z)\big) < \varepsilon$ for each $y, z \in G$.

A function $f\colon X \to Z$ (Z is a topological space) is said to be *quasicontinuous at* $x \in X$ ([10]) if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$.

A function f is *cliquish* (*quasicontinuous*) if it is cliquish (quasicontinuous) at each point.

Denote by $C(f)$ and $A(f)$ the set of all continuity and cliquishness points of a function $f\colon X \to Y$, respectively. The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ stand for the set of real, rational and natural numbers, respectively. For a subset A of a topological space denote by $\operatorname{Cl} A$, $\operatorname{Int} A$ and A^d the closure of A, the interior of A and the set of all accumulation points of A, respectively.

Let $f\colon X \to Y$ be a function.

The function $\omega_f\colon X \to \mathbb{R} \cup \{\infty\}$, given by the formula

$$\omega_f(x) = \inf\big\{d\big(f(U)\big) : U \text{ is a neighbourhood of } x\big\},$$

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where $d(A) = \sup\{d(y,z)\colon y,z \in A\})$, is said to be the oscillation of f. It is well-known that f is continuous at x if and only if $\omega_f(x) = 0$.

In [8] it is defined a function $q_f\colon X \to \mathbb{R} \cup \{\infty\}$ by

$$q_f(x) = \sup\{\inf\{\sup\{d\,(f(x), f(y)) : y \in V\}\},$$

where supremum is taken over all neighbourhoods U of x and infimum is taken over all nonempty open subsets $V \subset U$ and it is proved that f is quasicontinuous at x if and only if $q_f(x) = 0$. (Notice that in [8] it is assumed that X is an interval and $Y = \mathbb{R}$.)

Similar mapping we may define for cliquishness points.

DEFINITION 1. The mapping $k_f\colon X \to \mathbb{R} \cup \{\infty\}$, given by the formula

$$k_f(x) = \sup\{\inf\{d(f(V))\colon V \subset U\}\colon U \ni x\},$$

where supremum is taken over all neighbourhoods U of x and infimum is taken over all nonempty open subsets $V \subset U$, will be called the *quasioscillation of* f.

Some properties of the oscillation are investigated in [7]. In this paper we shall deal with some properties of the quasioscillation. We obtain some well-known results on the cliquishness points as corollary. Evidently

PROPOSITION 1. *A function $f\colon X \to Y$ is cliquish at $x \in X$ if and only if $k_f(x) = 0$.*

Let $\mathcal{T}$ be the usual topology on $\mathbb{R} \cup \{\infty\}$ (i.e. the neighbourhood base of ∞ is $\{(a,\infty)\}_{a\in\mathbb{R}}$ and if $\infty \notin A$, then $A \in \mathcal{T}$ iff A is open in the usual topology on $\mathbb{R}$). If f is locally bounded, then ω_f, q_f and k_f are real functions.

It is well-known that ω_f is upper semicontinuous ([6]). We shall show that if f is locally bounded, then the oscillation of f is cliquish.

LEMMA 1. *Let $g\colon X \to \mathbb{R}$ be an upper semicontinuous and locally bounded below function. Then g is cliquish.*

Proof. Let $x \in X$, $\varepsilon > 0$ and U be an open neighbourhood of x. We can assume that $f(y) \ge b$ for each $y \in U$ and some $b \in \mathbb{R}$. Denote $A_n = \{y \in U\colon g(y) < b + n\varepsilon\}$. Then A_n are open sets, $A_0 = \emptyset$ and $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$. Put $m = \min\{n \in \mathbb{N}\colon A_n \ne \emptyset\}$. Let $y,z \in A_m$. Then $g(y), g(z) \in (b + (m-1)\varepsilon, b + m\varepsilon)$ and therefore $|g(y) - g(z)| < \varepsilon$, i.e. g is cliquish at x. □

COROLLARY 1. *If $f\colon X \to Y$ is locally bounded, then ω_f is cliquish.*

Remark 1. The assumption "f is locally bounded below" in Lemma 1 cannot be omitted. (Let $X = \mathbb{Q} = \{q_1, q_2, \dots\}$ (one-to-one sequence) and $f(q_n) = -n$.) If X is Baire, then Lemma 1 follows from the fact that $C(\omega_f)$ is residual.

Theorem 1. *The quasioscillation is lower semicontinuous and quasicontinuous.*

Proof. Let $x \in X$.

Let $k_f(x) > a$. Then there is an open neighbourhood U of x such that $\inf\{d(f(V)) \colon V \subset U \text{ nonempty open}\} > a$. Then $k_f(y) > a$ for each $y \in U$ and therefore k_f is lower semicontinuous at x.

If $k_f(x) = \infty$, then the quasicontinuity at x follows from the lower semicontinuity.

Let $k_f(x) < \infty$, U be a neighbourhood of x and $\varepsilon > 0$. Then there is a neighbourhood $U_1 \subset U$ of x such that $k_f(y) > k_f(x) - \varepsilon$ for each $y \in U_1$. Hence there is an open nonempty set $V \subset U_1$ such that $d(f(V)) < k_f(x) + \varepsilon$. Then $k_f(y) < k_f(x) + \varepsilon$ for each $y \in V$. Hence $|k_f(x) - k_f(y)| < \varepsilon$ for each $y \in V$ and k_f is quasicontinuous at x. □

Corollary 2. (See [9].) *The set $A(f)$ is closed.*

We shall show that the properties in Theorem 1 characterize the quasioscillation. Recall that a topological space X is resolvable [5] if it is the union of two disjoint dense sets.

Theorem 2. *Let X be a resolvable space. Let $g \colon X \to \mathbb{R}$ be a nonnegative quasicontinuous lower semicontinuous function. Then there is a function $f \colon X \to \mathbb{R}$ such that $k_f = g$.*

Proof. Let $X = A \cup B$, where A, B are dense disjoint. Define $f \colon X \to \mathbb{R}$ as

$$f(x) = \begin{cases} g(x), & \text{if } x \in A, \\ 0, & \text{if } x \in B. \end{cases}$$

Let $x \in X$, $\varepsilon > 0$.

From the quasicontinuity of g for each neighbourhood U of x there is an open nonempty set $V \subset U$ such that $|g(x) - g(y)| < \varepsilon$ for each $y \in V$. This yields $0 \le f(y) \le g(y) < g(x) + \varepsilon$ and hence $d(f(V)) \le g(x) + \varepsilon$. From this $k_f(x) \le g(x) + \varepsilon$.

On the other hand, there is a neighbourhood U of x such that $g(y) > g(x) - \varepsilon$ for each $y \in U$. Let $V \subset U$ be an open nonempty set. Let $y \in A \cap V$, $z \in B \cap V$. Then $d(f(V)) \ge |f(y) - f(z)| = g(y) > g(x) - \varepsilon$. Hence $k_f(x) \ge g(x) - \varepsilon$. Therefore $k_f(x) = g(x)$. □

COROLLARY 3. (See [4], cf. also [9]). *Let X be a resolvable perfect normal topological space. Let $A \subset X$ be closed. Then there is $f\colon X \to \mathbb{R}$ such that $A = A(f)$.*

P r o o f. . Let $g\colon X \to [0,1]$ be a continuous function such that $g^{-1}(0) = A$ and let $f\colon X \to \mathbb{R}$ be such that $g = k_f$. □

Now we shall investigate a relation between the oscillation and the quasioscillation.

THEOREM 3. *Let $f\colon X \to Y$. Then we have*

a) $k_f \le \omega_f$, $q_f \le \omega_f$, $k_f \le 2q_f$;

b) $C(f) \subset C(\omega_f) \cap C(k_f) \cap C(q_f)$;

c) $k_f(x) = \omega_f(x)$ *if and only if* $x \in C(\omega_f)$;

d) $C(\omega_f) \subset C(k_f)$;

e) *if* $x \in C(\omega_f)$, *then* $\omega_f(x) \le 2q_f(x)$;

f) $\{x \in X\colon \omega_f(x) \ne k_f(x)\}$ *is of the first category;*

g) $k_f(x) = \liminf\limits_{u \to x} \omega_f(u)$.

P r o o f.

a) is easy.

b) follows easy from a).

c) Necessity. Let $k_f(x) = \omega_f(x)$. Let $\omega_f(x) > a$. Then there is a neighbourhood U of x such that $k_f(x) > a$ for each $y \in U$. By a) $\omega_f(y) > a$. Therefore ω_f is lower semicontinuous and $x \in C(\omega_f)$.

Sufficiency. Let $k_f(x) \ne \omega_f(x)$. Then by a) there is α such that $k_f(x) < \alpha < \omega_f(x)$. Since $x \in C(\omega_f)$, there is a neighbourhood U of x such that $\omega_f(y) > \alpha$ for each $u \in U$. Since $k_f(x) < \alpha$, there is a nonempty open set $V \subset U$ such that $d\big(f(V)\big) < \alpha$. Hence $\omega_f(y) < \alpha$ for each $y \in V$, a contradiction.

d) Let $x \in C(\omega_f)$ and $k_f(x) < a$. Then by c) $\omega_f(x) < a$ and there is a neighbourhood U of x such that $\omega_f(y) < a$ for each $y \in U$. Further by a) $k_f(y) < a$ and hence k_f is upper semicontinuous at x. By Theorem 1 $x \in C(k_f)$.

e) follows from c) and a).

f) follows from c).

g) Let $\liminf\limits_{u \to x} \omega_f(u) < \alpha$. Then for each open neighbourhood U of x there is $u \in U$ such that $\omega_f(u) < \alpha$ and thus there is an open neighbourhood $V \subset U$ of u such that $d\big(f(V)\big) < \alpha$. Therefore $k_f(x) \le \alpha$.

Let $\liminf_{u\to x} \omega_f(u) > \alpha$. Then there is a neighbourhood U of x such that $\omega_f(u) > \alpha$ for each $u \in U$. Let $V \subset U$ be nonempty open. Then V is a neighbourhood of some $u \in U$ and hence $d(f(V)) > \alpha$. Therefore $k_f(x) > \alpha$. This yields $k_f(x) = \liminf_{u\to x} \omega_f(u)$. □

COROLLARY 4. (See [11].) *The set $A(f) - C(f)$ is of the first category.*

LEMMA 2. *Let $h\colon X \to \mathbb{R}$ be lower semicontinuous, $g\colon X \to \mathbb{R}$ be upper semicontinuous and $\{x \in X\colon h(x) = g(x)\}$ be dense in X. Then $h(x) \le g(x)$ for each $x \in X$.*

Proof. Suppose that $h(x) > g(x)$ for some $x \in X$. Denote $\eta = \frac{1}{2}\big(h(x) - g(x)\big) > 0$. Then there is a neighbourhood U of x such that $h(y) > h(x) - \eta$ and $g(y) < g(x) + \eta$ for each $y \in U$. Let $z \in U$ be such that $g(z) = h(z)$. Then $g(z) = h(z) > h(x) - \eta = g(x) + \eta > g(z)$, a contradiction. □

THEOREM 4. *Let X be a Baire first countable space. Let $h, g\colon X \to \mathbb{R}$ be functions. Then there is a function $f\colon X \to \mathbb{R}$ such that $\omega_f = g$ and $k_f = h$ if and only if*

(1) *h, g are nonnegative,*
(2) *g is upper semi-continuous,*
(3) *h is lower semi-continuous and quasicontinuous,*
(4) *$h(x) = g(x)$ for each $x \in C(g)$,*
(5) *$g(x) = 0$ for each isolated point $x \in X$.*

Proof. Necessity. According to Theorems 1 and 3.

Sufficiency. Denote $Z = X - \mathrm{Cl}(X - X^d) \subset X^d$. Since X is Baire, the set $C(g)$ is dense in X. The space $Z \cap C(g)$ is first countable without isolated points and hence by [5] $Z \cap C(g) = A \cup B$, where A and B are disjoint and dense in $Z \cap C(g)$ (and therefore also in Z). Define a function $f\colon X \to \mathbb{R}$ as

$$f(x) = \begin{cases} 0, & \text{if } x \in A, \\ g(x), & \text{otherwise.} \end{cases}$$

First we shall show that $\omega_f = g$.

Let $x \in X$, $\varepsilon > 0$. Then by (1) and (2) there is a neighbourhood U of x such that $0 \le f(y) \le g(y) < g(x) + \varepsilon$ for each $y \in U$. Hence $d(f(U)) \le g(x) + \varepsilon$ and $\omega_f(x) \le g(x) + \varepsilon$.

If $x \in A$, then $x \in C(g)$ and there is a neighbourhood G of x such that $g(y) > g(x) - \varepsilon$ for each $y \in G$. For $z \in G \cap B$ we have $f(z) = g(z) > g(x) - \varepsilon$ and hence $\omega_f(x) \ge g(x) - \varepsilon$.

If $x \in B \cup (Z - C(g))$, then $\omega_f(x) \geq g(x)$ with respect to the density of A. If $x \in \mathrm{Cl}(X - X^d)$, then evidently $\omega_f(x) \geq g(x)$. Therefore $\omega_f = g$.

Now we shall show that $k_f = h$. Let $x \in X, \varepsilon > 0$.

Let $x \in \mathrm{Cl}(X - X^d)$. Then by (3) there is a neighbourhood U of x such that $h(x) < h(y) + \varepsilon$ for each $y \in U$. Then for $z \in U \cap (X - X^d)$ we have $h(x) < h(z) + \varepsilon = \varepsilon$. Therefore according to Lemma 2 and (5) we have $k_f(x) = 0 = h(x)$.

Let $x \in Z$. Then there is a neighbourhood $U \subset Z$ of x such that $h(y) > h(x) - \varepsilon$ for each $y \in U$. Let $V \subset U$ be nonempty open. Let $y \in A \cap V$, $z \in B \cap V$. Then according to Lemma 2 we have $d(f(V)) \geq |f(y) - f(z)| = g(y) \geq h(y) > h(x) - \varepsilon$ and therefore $k_f(x) \geq h(x) - \varepsilon$.

Now let W be an arbitrary neighbourhood of x. By (3) there is a nonempty open set $V \subset W$ such that $h(y) < h(x) + \frac{\varepsilon}{2}$ for each $y \in V$. Let $z \in V \cap C(g)$. Then there is an open neighbourhood $G \subset V$ of z such that $|g(u) - g(z)| < \frac{\varepsilon}{4}$ for each $u \in G$. From the quasicontinuity of h at z there is an open nonempty set $H \subset G$ such that $|h(u) - h(z)| < \frac{\varepsilon}{4}$ for each $u \in H$. Since $h(z) = g(z)$ by (3) for each $u \in H$ we have $|g(u) - h(u)| \leq |g(u) - g(z)| + |h(z) - h(u)| < \frac{\varepsilon}{2}$ and therefore

$$0 \leq f(u) \leq g(u) < h(u) + \frac{\varepsilon}{2} < h(x) + \varepsilon .$$

Hence $d(f(H)) \leq h(x) + \varepsilon$ and $k_f(x) \leq h(x) + \varepsilon$. Therefore for each $x \in X$ we have $k_f(x) = h(x)$. □

R e m a r k 2. The assumption "X is Baire" cannot be omitted. (If $X = \mathbb{Q} = \{q_1, q_2, \dots\}$ (one-to-one sequence), $g(q_n) = \frac{1}{n}$ and $h(q_n) = 1$ for each $n \in \mathbb{N}$, then g, h satisfy (1)–(5), however there is no function $f\colon X \to \mathbb{R}$ such that $k_f = h$ and $\omega_f = g$.)

COROLLARY 5. *Let X be a Baire first countable perfect normal space. Let $A, C \subset X$. Then there is a function $f\colon X \to \mathbb{R}$ such that $C = C(f)$ and $A = A(f)$ if and only if C is G_δ and contains all isolated points of X, A is closed, $C \subset A$ and $A - C$ is of the first category.*

P r o o f . Let $h\colon X \to [0,1]$ be a continuous function such that $h^{-1}(0) = A$. Let F_n be closed nowhere dense sets such that $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$ and $A - C = \bigcup_{n=1}^{\infty} F_n$, $F_0 = \emptyset$. Let $g(x) = \frac{1}{n}$ for $x \in F_n - F_{n-1}$ and $g(x) = h(x)$ otherwise. Then it is easy to verify that h, g satisfy (1)–(5), $h^{-1}(0) = A$ and $g^{-1}(0) = C$. □

(More detailed on the sets $A(f)$ and $C(f)$ see in [4].)

Lemma 3. *Let A be an open set, let $f\colon X \to Y$. Then for each $x \in A$ we have*

$$\max\{|k_f(x) - k_g(x)|, |\omega_f(x) - \omega_g(x)|, |q_f(x) - q_g(x)|\} \leq \\ \leq 2\sup\{d(f(x), g(x))\colon x \in A\}.$$

Proof. Suppose that there is $x \in A$ such that $k_g(x) - k_f(x) > 2\sup\{d(f(x), g(x))\colon x \in A\} = a$. Then there is α such that $k_g(x) > \alpha > k_f(x) + a$ and hence there is a neighbourhood $U \subset A$ of x such that $d(g(V)) > \alpha$ for each nonempty open $V \subset U$. Since $k_f(x) < \alpha - a$, there is a nonempty open $V \subset U$ such that $d(f(V)) < \alpha - a$. Then there are $y, z \in V$ such that $d(g(y), g(z)) > \alpha$ and hence

$$d(f(y), f(z)) + a \leq d(f(V)) + a < \alpha < d(g(y), g(z)) \leq \\ d(g(y), f(y)) + d(f(y), f(z)) + d(f(z), g(z)) \leq d(f(y), f(z)) + a,$$

a contradiction. Similar for q_f and ω_f. □

It is easy to see that if a sequence $(f_n)_n$ uniformly converges to f, then $(\omega_{f_n})_n$ uniformly converges to ω_f. In [8] it is shown than the uniform convergence of $(f_n)_n$ to f implies the uniform convergence $(q_{f_n})_n$ to q_f. From Lemma 3 we see

Proposition 2. *If a sequence $(f_n)_n$, $f_n\colon X \to Y$ uniformly converges to $f\colon X \to Y$, then $(k_{f_n})_n$ uniformly converges to k_f.*

We recall that a sequence $(f_n)_n$, $f_n\colon X \to Y$ quasiuniformly converges to $f\colon X \to Y$, if it pointwise converges to f and

$$\forall \varepsilon > 0\ \forall m \in \mathbb{N}\ \exists p \in \mathbb{N}\ \forall x \in X\colon \\ \min\{d(f_{m+1}(x), f(x)), \dots, d(f_{m+p}(x), f(x))\} < \varepsilon.$$

In [1] it is shown that if $(f_n)_n$ quasiuniformly converges to f, then $\omega_f(x) \leq 2\limsup_{n\to\infty} \omega_{f_n}(x)$ for each $x \in X$.

Proposition 3. *Let $(f_n)_n$ quasiuniformly converge to f. Then the sets $\{x \in X\colon k_f(x) > 2 \cdot \limsup_{n\to\infty} k_{f_n}(x)\}$ and $\{x \in X\colon q_f(x) > 4 \cdot \limsup_{n\to\infty} q_{f_n}(x)\}$ are of the first category in X.*

Proof. By [1] for each $x \in X$ we have $\omega_f(x) \leq 2 \cdot \limsup_{n\to\infty} \omega_{f_n}(x)$. By Theorem 3 g) the set $A_n = \{x \in X\colon k_{f_n}(x) \neq \omega_{f_n}(x)\}$ is of the first category for each $n \in \mathbb{N}$. Hence the set

$$A = \bigcup_{n=1}^{\infty} A_n = \{x \in X\colon \exists n \in \mathbb{N};\ \omega_{f_n}(x) \neq k_{f_n}(x)\}$$

is of the first category. For each $x \in X - A$ we have $k_f(x) \leq \omega_f(x) \leq 2 \cdot \limsup_{n\to\infty} \omega_{f_n}(x) = 2 \cdot \limsup_{n\to\infty} k_{f_n}(x)$. Hence $\{x \in X\colon k_f(x) > 2 \cdot \limsup_{n\to\infty} k_{f_n}(x)\} \subset A$ is of the first category.

Similarly, by Theorem 3 e) the set $B_n = \{x \in X\colon \omega_{f_n}(x) > 2q_{f_n}(x)\}$ is of the first category and hence $\{x \in X\colon q_f(x) > 4 \cdot \limsup_{n\to\infty} q_{f_n}(x)\}$ is of the first category.

Corollary 6. (See [3].) *If X is a Baire space, then the quasiuniform limit of cliquish functions is cliquish.*

(In [2] it is shown that this is not true for an arbitrary topological space X.)

Proposition 4. *Let $f\colon X \to \mathbb{R}$. Then $k_f = f$ if and only if $f(x) = 0$ for each $x \in X$.*

Proof. Let $k_f = f$. Then by Theorem 1 f is quasicontinuous and hence cliquish. Hence $k_f(x) = f(x) = 0$ for each $x \in X$. □

Remark 3. Let $f\colon X \to \mathbb{R}$ be locally bounded. Denote $\omega^1(f) = \omega_f$ and $\omega^{n+1}(f) = \omega_{\omega^n(f)}$ for $n > 1$. Similarly for $k^n(f)$ and $q^n(f)$.

In [7] it is shown that $\omega^n = \omega^2$ for $n \geq 2$. From Proposition 4 we obtain $k^n = k^2 (= 0)$ for $n \geq 2$. Similar assertion is not true for q^n. (Let $(n, n+1) = \bigcup_{i=1}^{2n+1} A_i^n$, where A_i^n are disjoint dense in $(n, n+1)$. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = i$ for $x \in A_i^n$ and $f(x) = 0$ otherwise. Then $q^n(f) \neq q^{n+1}(f)$ for any $n \in \mathbb{N}$.)

The set $\mathbb{R}^X$ can be considered as a topological space with the topology of uniform convergence. This topology is metrized by the metric $\rho(f,g) = \min\{1, \sup\{|f(x) - g(x)|\colon x \in X\}\}$. Denote $M(X) = \{f\colon X \to \mathbb{R}\colon f$ is locally bounded$\}$. Further denote $S(k) = \{f\colon X \to \mathbb{R}\colon k_f$ is continuous$\}$; similarly $S(\omega)$ and $S(q)$. In [7] it is shown that if X is a metric space with $X^d \neq \emptyset$, then $S(\omega) \cap M(X)$ is perfect and nowhere dense in $M(X)$.

Theorem 5.

(i) *The sets* $S(\omega)$, $S(k)$ *and* $S(q)$ *are perfect in* $\mathbb{R}^X$.

(ii) *If* X *is* T_1 *first countable and* $X^d \neq \emptyset$, *then* $S(\omega) \cap M(X)$ *is nowhere dense in* $M(X)$.

(iii) *If* X *is* T_2 *first countable and* $X - \mathrm{Cl}(X - X^d) \neq \emptyset$, *then* $S(k) \cap M(X)$ *is nowhere dense in* $M(X)$.

(iv) *If* X *is* T_1 *and* $X^d \neq \emptyset$, *then* $S(q) \cap M(X)$ *is nowhere dense in* $M(X)$.

(v) *The sets* $S(\omega)$, $S(k)$ *and* $S(q)$ *need not be nowhere dense in* $\mathbb{R}^X$.

First we shall prove the following lemma

Lemma 4. *Let* X *be a Hausdorff first countable topological space without isolated points. Let* $f\colon X \to \mathbb{R}$, $x_0 \in X$ *and* $k_f(x_0) < b$. *Then there are sequences* $(A_n)_n$, $(B_n)_n$ *of open disjoint sets such that* $d(f(A_n)) < b$ *and* $d(f(B_n)) < b$ *for each* $n \in \mathbb{N}$ *and for each neighbourhood* U *of* x_0 *there is* $n_0 \in \mathbb{N}$ *such that* $A_n \cup B_n \subset U$ *for each* $n \geq n_0$.

Proof. Let $(U_n)n$ be a decreasing neighbourhood base of x_0. Then there is a nonempty open set $G_1 \subset U_1$ such that $d(f(G_1)) < b$. Since X is T_2 without isolated points, there is $n_1 \in \mathbb{N}$ such that $H_1 = G_1 - \mathrm{Cl}\, U_{n_1} \neq \emptyset$. Further there is a nonempty open set $G_2 \subset U_{n_1}$ such that $d(f(G_2)) < b$ and $n_2 \in \mathbb{N}$ such that $H_2 = G_2 - \mathrm{Cl}\, U_{n_2} \neq \emptyset$. In this way we construct a sequence $(H_n)_n$ of nonempty open disjoint sets such that $d(f(H_n)) < b$ for each $n \in \mathbb{N}$ and for each neighbourhood U of x_0 there is $n_0 \in \mathbb{N}$ such that $H_n \subset U$ for each $n \geq n_0$. Since X is T_2 without isolated points, there are nonempty open disjoint sets A_n, B_n such that $A_n \cup B_n \subset H_n$. □

Proof of Theorem 5.

(i) By Lemma 3 the sets $S(\omega)$, $S(k)$ and $S(q)$ are closed in $\mathbb{R}^X$. Since $k_f = k_{f+c}$ for each $c \in \mathbb{R}$ (similarly for ω_f and q_f), they are dense in itself and hence perfect.

(ii) The proof is the same as in [7].

(iii) Let $f\colon X \to \mathbb{R}$, $0 < \varepsilon < 1$. We may assume that k_f is continuous. Since $X - \mathrm{Cl}(X - X^d) \neq \emptyset$, there is a nonempty open set H in X without isolated points. Let $x_0 \in H$. Now we have two possibilities:

I. $k_f(x_0) = 0$.

Since H is first countable without isolated points , according to [5] there are dense disjoint sets C, D in H such that $H = C \cup D$. Put $b = \frac{\varepsilon}{4}$. Let A_n, B_n

be sets from Lemma 4. Define $g\colon X \to \mathbb{R}$,

$$g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & \text{if } x \in B_n \cap D \text{ for some } n \in \mathbb{N}, \\ f(x), & \text{otherwise.} \end{cases}$$

Then $\rho(f,g) < \varepsilon$. Let U be a neighbourhood of x_0. Since k_f is continuous, there is a neighbourhood $U_0 \subset H \cap U$ of x_0 such that $k_f(y) < \frac{\varepsilon}{8}$ for each $y \in U_0$. Let n be such that $A_n \cup B_n \subset U_0$.
Let $y \in A_n$. Then $k_g(y) = k_f(y) < \frac{\varepsilon}{8}$.
Let $z \in B_n$. Let $G \subset B_n$ be nonempty open. Let $u \in G \cap D, z \in G \cap C$. Then $|f(u) - f(v)| < \frac{\varepsilon}{4}$ and hence $g(u) - g(v) > \frac{\varepsilon}{4}$. This yields $k_g(z) > \frac{\varepsilon}{4}$ and hence $k_g(z) - k_g(y) > \frac{\varepsilon}{8}$. Therefore k_g is discontinuous at x_0.

II. $k_f(x_0) = s > 0$.
Put

$$a = \max\left\{s - \frac{\varepsilon}{8}, \frac{3}{4}s\right\}, \quad b = \min\left\{\frac{9}{8}s, s + \frac{\varepsilon}{8}\right\}.$$

Then $0 < a < s < b$. Let A_n, B_n be sets from Lemma 4. Since $k_f(x_0) > a$, there is a neighbourhood $U_0 \subset H$ of x_0 such that $d(f(G)) > a$ for each nonempty open $G \subset U_0$. Let $n_0 \in \mathbb{N}$ be such that $A_n \cup B_n \subset U_0$ for each $n \geq n_0$.

Let $n \geq n_0$ and $G \subset B_n$ be nonempty open. Since $d(f(G)) > a$, there are $y_G^n, z_G^n \in G$ such that $f(z_G^n) - f(y_G^n) > a$. We shall show that $y_V^n \neq z_W^n$ for all nonempty open $V, W \subset B_n$.

Suppose that there are nonempty open sets $V, W \subset B_n$ such that $y_V^n = z_W^n$. Then $V \cap W$ is a nonempty open subset of B_n and $\max\{f(z_{V\cap W}^n) - f(y_V^n), f(y_V^n) - f(y_{V\cap W}^n)\} > \frac{a}{2}$. Let e.g. $f(z_{V\cap W}^n) - f(y_V^n) > \frac{a}{2}$. Then

$$f(z_{V\cap W}^n) - f(y_W^n) = f(z_{V\cap W}^n) - f(z_W^n) + f(z_W^n) - f(y_W^n) >$$
$$> \frac{a}{2} + a = \frac{3}{2}a \geq \frac{9}{8}s \geq b.$$

However $d(f(V \cap W)) < b$, a contradiction.

Define a function $g\colon X \to \mathbb{R}$,

$$g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & \text{if } x = z_G^n \text{ for some } n \geq n_0 \text{ and for some} \\ & \text{nonempty open } G \subset B_n, \\ f(x), & \text{otherwise.} \end{cases}$$

Then $\rho(f,g) < \varepsilon$. Let $U \subset H$ be a neighbourhood of x_0. Then there is $m \geq n_0$ such that $A_m \cup B_m \subset U$.
Let $y \in A_m$. Then $k_g(y) = k_f(y) \leq b$.

Let $z \in B_m$. Then for each nonempty open set $G \subset B_m$ we have $g(z_G^m) - g(y_G^m) > a + \frac{\varepsilon}{2}$ and hence $k_g(z) \geq a + \frac{\varepsilon}{2}$. Therefore $k_g(z) - k_g(y) \geq a + \frac{\varepsilon}{2} - b \geq \frac{\varepsilon}{4}$ and hence k_g is discontinuous at x_0.

(iv) Let $f\colon X \to \mathbb{R}$, $0 < \varepsilon < 1$ and q_f be continuous. Let $x_0 \in X^d$ and define $g\colon X \to \mathbb{R}$,

$$g(x) = \begin{cases} f(x) + \frac{\varepsilon}{2}, & \text{if } f(x) \geq f(x_0) \text{ and } x \neq x_0, \\ f(x) - \frac{\varepsilon}{2}, & \text{if } f(x) < f(x_0), \\ f(x_0), & \text{if } x = x_0. \end{cases}$$

Then it is not difficult to verify that $q_g(x_0) = q_f(x_0) + \frac{\varepsilon}{2}$ and $q_g(x)$ is equal to $q_f(x) + \varepsilon$ or $q_f(x)$. This yields that q_g is discontinuous at x_0.

(v) Let $X = \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ take on every real value in every interval. Then $k_g(x) = q_g(x) = \omega_g(x) = \infty$ for each g with $\rho(f,g) < 1$. □

Remark 4.

1. If $X^d = \emptyset$, then every $f\colon X \to \mathbb{R}$ is continuous and hence $S(\omega)$ and $S(q)$ are not nowhere dense.

2. If $X - \mathrm{Cl}(X - X^d) = \emptyset$, then every $f\colon X \to \mathbb{R}$ is cliquish and hence $S(k)$ is not nowhere dense.

3. The assumption "X is T_1" in (ii) and (iv) cannot be omitted. (Let $X = \{a,b\}$, $\mathcal{T} = \{\emptyset, X\}$. Then $S(\omega) = S(q) = \mathbb{R}^X$.)

4. The condition "X is T_2" in (iii) cannot be replaced by "X is T_1". (Let $X = \mathbb{N}$ with the cofinite topology. Then X is T_1 first countable without isolated points. It is easy to see that the quasioscillation of each function $f\colon X \to \mathbb{R}$ is constant.)

REFERENCES

[1] BORSÍK, J.: *On oscillation of limit functions*, Acta Math. Univ. Comenian **LX** (1991), 211–217.

[2] DOBOŠ, J.: *Some generalizations of the notion of continuity and quasi-uniform convergence*, Časopis pěst. mat. **106** (1981), 431–434.

[3] DOBOŠ, J.—ŠALÁT, T.: *Cliquish functions, Riemann integrable functions and quasi-uniform convergence*, Acta Math. Univ. Comenian **XL–XLI** (1982), 219–223.

[4] EWERT, J.—LIPIŃSKI, J. S.: *On points of continuity, quasicontinuity and cliquishness of maps*, Colloq. Math. Soc. János Bolyai, 41. Top. and Appl., Eger (Hungary), 1983, 269–281.

[5] HEWITT, E.: *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.

[6] KELLEY, J. L.: *General Topology*, Van Nostrand, New York, 1955.

[7] KOSTYRKO, P.: *Some properties of oscillation*, Math. Slovaca **30** (1980), 157–162.

[8] KOSTYRKO, P.: *Quasicontinuity and some classes of Baire 1 functions*, Comment. Math. Univ. Carolin **29** (1988), 601–609.

[9] LIPIŃSKI, J. S.—ŠALÁT, T.: *On the points of quasicontinuity and cliquishness of functions*, Czechoslovak Math. J. **21** (1971), 484–489.

[10] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exchange **14** (1988–89), 259–306.

[11] NEUBRUNNOVÁ, A.: *On quasicontinuous and cliquish functions*, Časopis pěst. mat. **99** (1974), 109–114.

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